# DNS Modeling of the VKS Experiment

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## velocity and magnetic field boundary conditions

Sebastian Kreuzahler

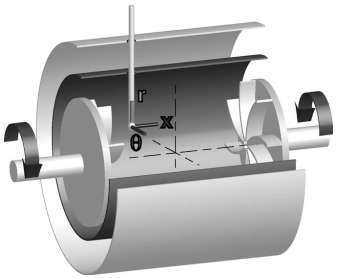
Theoretische Physik I
Ruhr-Universität Bochum

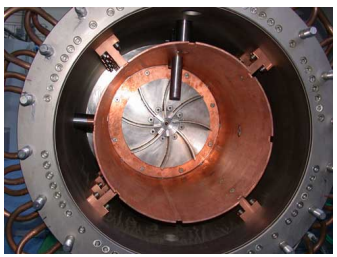
December 2013

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# VKS experiment - a fluid-solid-dynamo



Von Karman sodium experiment (VKS) in Cadarache:

- cylindrical vessel made of copper, filled with liquid sodium, driven by two soft-iron impellers
- successful observation of the dynamo instability in the laboratory

Impeller disks and blades must be soft iron. Occurance of dynamo instability depends on the **material** of the impellers!
$\rightarrow$ "fluid-solid dynamo"

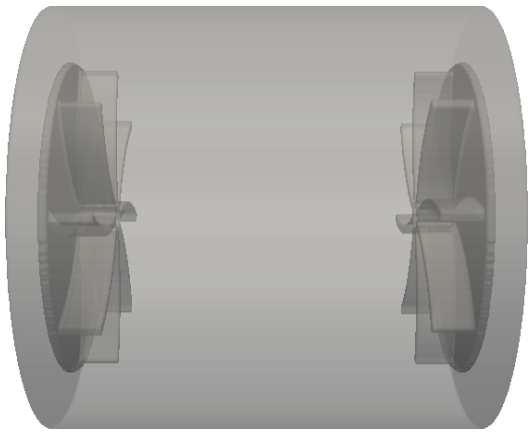
# Configuration (experiment and simulation)

Simulated impellers $\simeq$ experimental TM73 (VKS impeller):

Number of blades: 8
Disc radius $R_{disc} = 0.75 R_c$
Blade curvature radius $C = 0.9 R_c$
Curvature angle $\alpha = \arcsin(R_{disc}/2C) \approx 0.43$
Blade height $h_b = 0.2 R_c$

At resolution $256^3$ the maximum Reynolds number is:
$Re_{max} = \frac{VL}{\nu_{min}} = \frac{\Omega R_{disc} R_c}{\nu_{min}} \approx 2000$ (experiment: up to $10^7$)





# Equations - fluid

Incompressible fluids (Navier-Stokes):

$$\partial_t \boldsymbol{u} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{u} - \nabla p + \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0$$

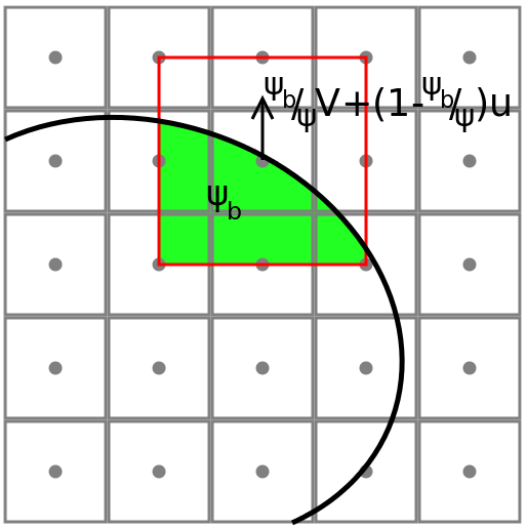
Boundary condition for the velocity field:

$$\boldsymbol{u}|_{boundary} = \boldsymbol{V}_{boundary}$$

Equations are solved with a pseudospectral code using FFT and penalty method for boundary.

Typical Reynolds numbers $Re = LU/\nu$:
Simulation $10^2$ - $10^3$, experiment $10^5$ - $10^7$, real dynamos $10^9$ - $10^{12}$

# Penalty




Problem: arbitrarily shaped **moving** boundary surface
$\Rightarrow$ solution: introduction of an artifical force, preserving the boundary condition

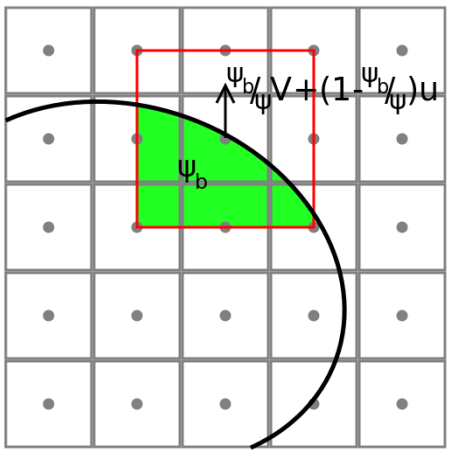
$$\boldsymbol{u}|_{boundary} = \boldsymbol{V}_{boundary}$$

direct forcing: $\boldsymbol{f}^{(m)} = -\chi(\boldsymbol{r},t)\left\{\mathcal{L}\boldsymbol{u}^{(m)} + \frac{1}{\Delta t}\left(\boldsymbol{u}^{(m)} - \boldsymbol{V}^{(m+1)}\right)\right\}$
with characteristic function $\chi$ and RHS $\mathcal{L}\boldsymbol{u}$
interpolation at the boundary with **volume fraction** $\Psi_b/\Psi$:

$$\overline{\chi}(\boldsymbol{r},t) = \frac{1}{2^3\Delta x\Delta y\Delta z}\int_{-\Delta x}^{\Delta x}\int_{-\Delta y}^{\Delta y}\int_{-\Delta z}^{\Delta z}\chi(\boldsymbol{r}+\boldsymbol{r}^*,t)d^3r^*$$

see: Fadlun et al., Journal of Computational Physics 161 (2000), pp. 35

# Impeller rotation




Previous method: rotation in physical space

Initial step: Abstract definition of simple geometric bodies
Procedure in each step: Calculate mask function $\overline{\chi}$

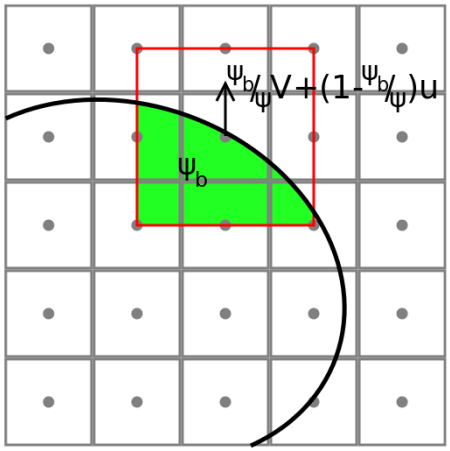
A lot faster: rotation in Fourier space

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tan\theta/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan\theta/2 \\ 0 & 1 \end{pmatrix} = \mathcal{S}_x(\theta)\mathcal{S}_y(\theta)\mathcal{S}_x(\theta)$$

E.g. first operation (skewing in x direction):

$$\overline{\chi}^*(\boldsymbol{x}) = \overline{\chi}(\mathcal{S}_x(\theta)\boldsymbol{x}) = \mathcal{F}_x^{-1}[\exp(-ik_x y \tan\theta/2)\mathcal{F}_x[\overline{\chi}(\boldsymbol{x})]]$$

# Higher order interpolation

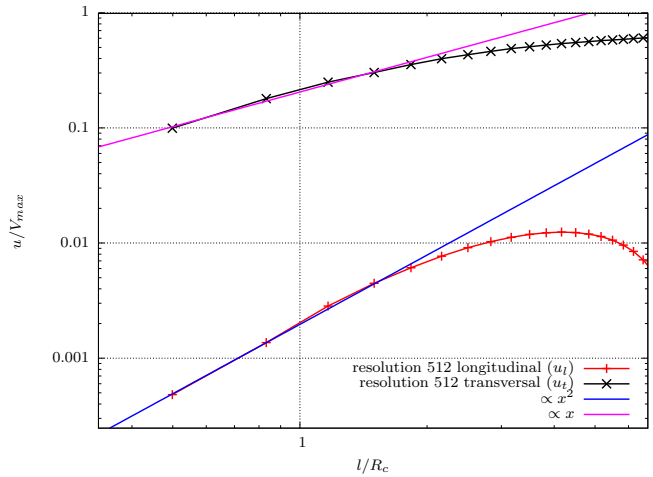



Volume fraction is only first order.
According to Fadlun et al. linear interpolation is more accurate.

But: Transversal and longitudinal velocity behave differently.

Wall in $xy$-plane: $\nabla \cdot \boldsymbol{u} = \partial_x u_x + \partial_y u_y + \partial_z u_z = 0$
Parallel components in lowest order: $u_{x,y} \propto z$
Since variation of these in tangential direction is zero, perpendicular component holds:
$\partial_z u_z = 0 \Rightarrow u_z \propto z^2$

Using volume fraction, the velocity field adapts to these conditions.
High order interpolation must consider this.
Question: How to interpolate mirror points?



# Equations - MHD with solid matter

Incompressible liquid metals + solid (modified MHD):

$$\partial_t \boldsymbol{u} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{u} - \nabla p + \boldsymbol{j}\times\boldsymbol{B} + \boldsymbol{f}$$

$$\partial_t \boldsymbol{B} = \nabla\times\left(\boldsymbol{u}\times\boldsymbol{B} - \eta\nabla\times\frac{\boldsymbol{B}}{\mu}\right) - \nabla\Pi + \boldsymbol{g}$$

$$\nabla\cdot\boldsymbol{B} = 0 \qquad \nabla\cdot\boldsymbol{u} = 0$$

+ velocity field boundary conditions (see above)
+ magnetic field boundary conditions (outside)
+ magnetic field jump conditions (impellers)

Typical magnetic Reynolds numbers $Rm = LU/\eta$:
Simulation $10^2$ - $10^3$, Experiment $\approx 50$, real dynamos $10^2$ - $10^6$
Prandtl numbers $Pm = Rm/Re$:
Simulation $0.1$ - $10$, Experiment $10^{-5}$, real dynamos $10^{-6}$ - $10^{-7}$

# Treatment of solids

Outer boundary: either periodic (no penalization) or vanishing tangential fields (VTF) via penalization
$B_t = 0$ or $\boldsymbol{B}|_{boundary} = B_n \boldsymbol{e}_n|_{boundary}$

Impellers: jump conditions are not enforced, only modified equation is solved with locally different resistivity and permeability:
$\eta = \eta_0 + \delta\eta \cdot \chi_{imp}$, $1/\mu = \alpha = 1 + \delta\alpha \cdot \chi_{imp}$
Jump conditions:

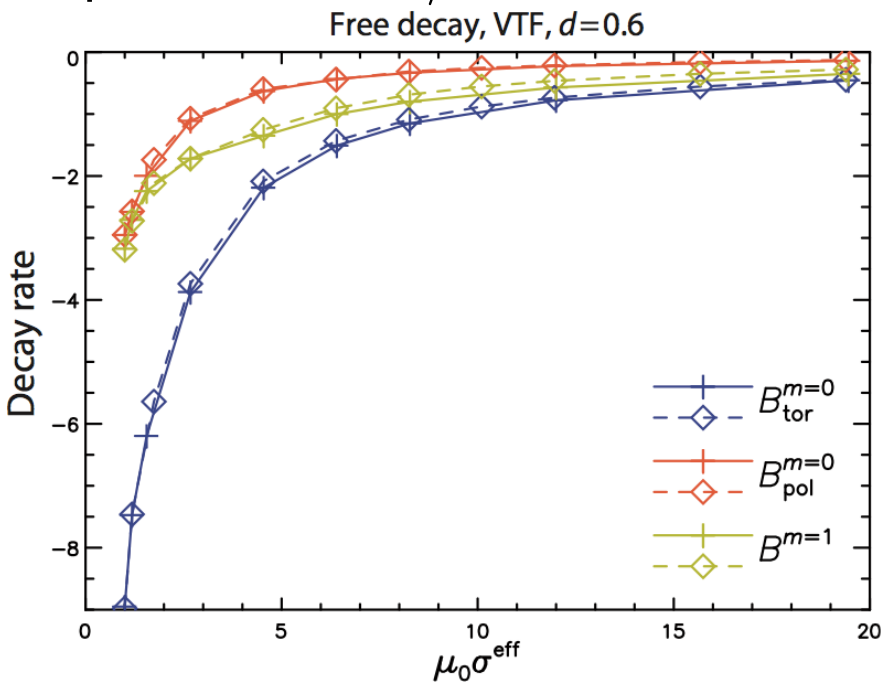
$$\boldsymbol{e}_n \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0, \quad \boldsymbol{e}_n \times \left( \frac{\boldsymbol{B}_1}{\mu_1} - \frac{\boldsymbol{B}_2}{\mu_2} \right) = 0, \qquad \text{jump in } \mu$$

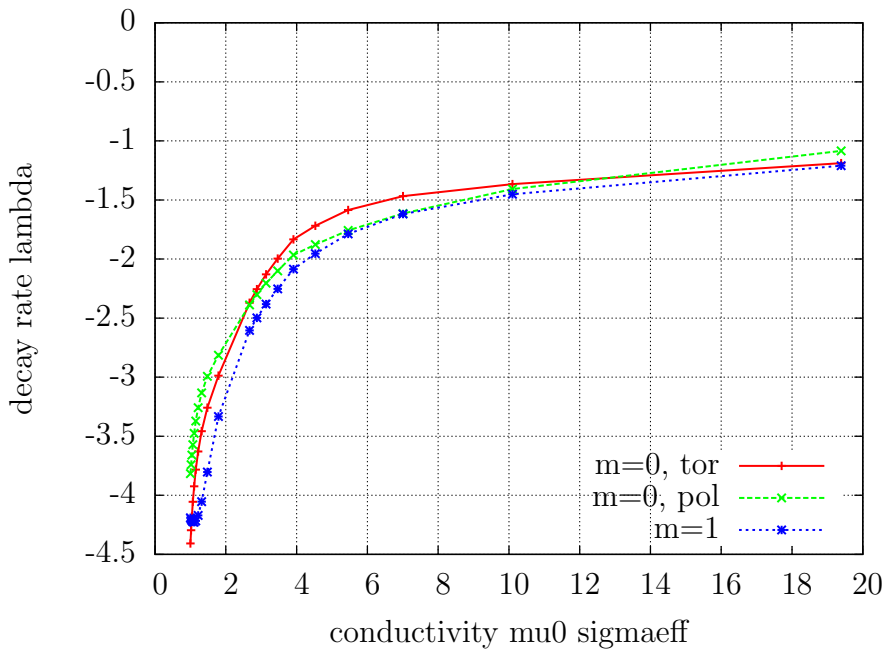
$$\boldsymbol{e}_n \cdot (\boldsymbol{j}_1 - \boldsymbol{j}_2) = 0, \quad \boldsymbol{e}_n \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0 \qquad \text{jump in } \eta$$

# Benchmark - decay of magnetic field

Configuration: Two thick disks inside cylinder, VTF condition, $\boldsymbol{u} = 0$

Decay of magnetic field components for different disk permeabilities/conductivities?



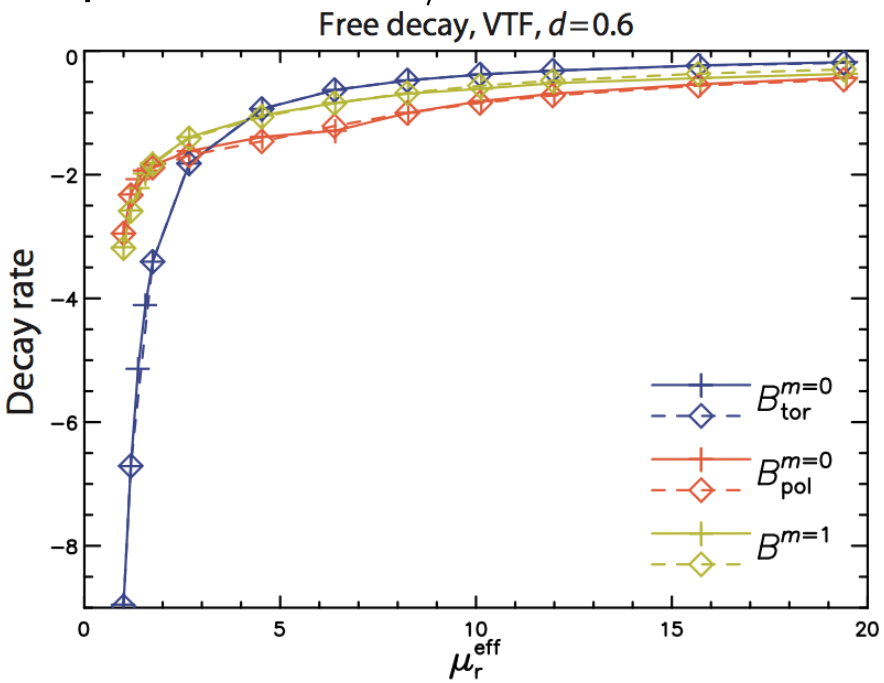



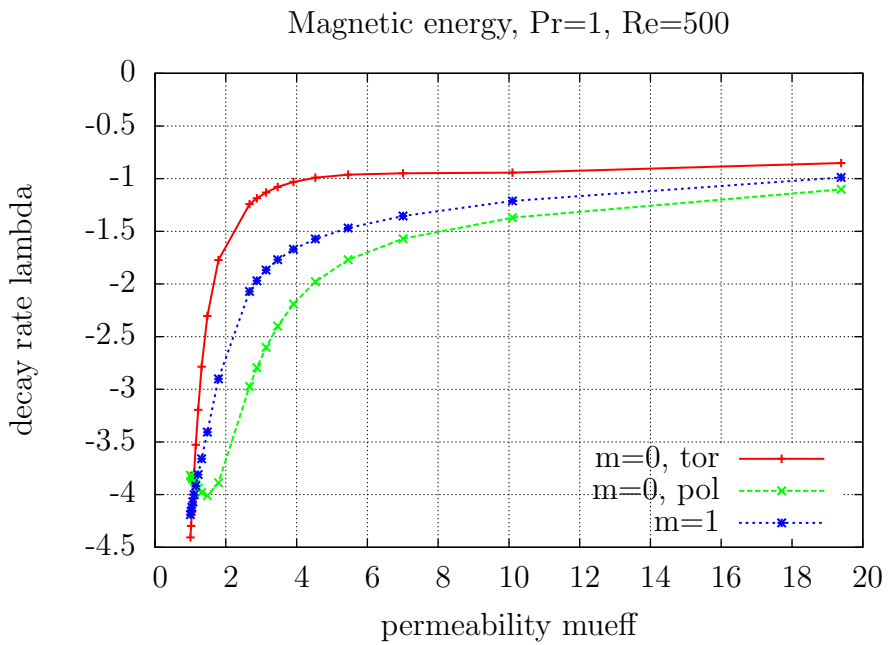

Conductivity

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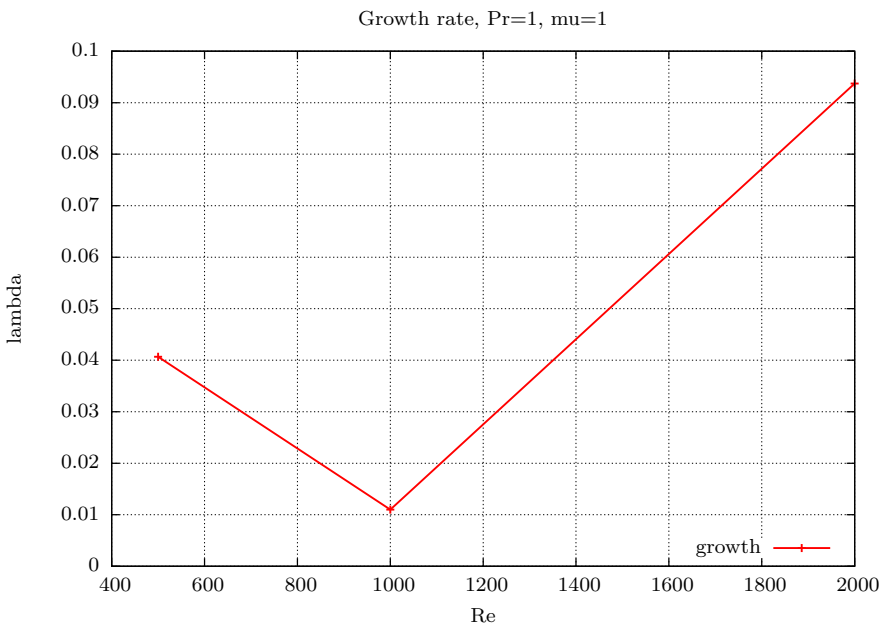


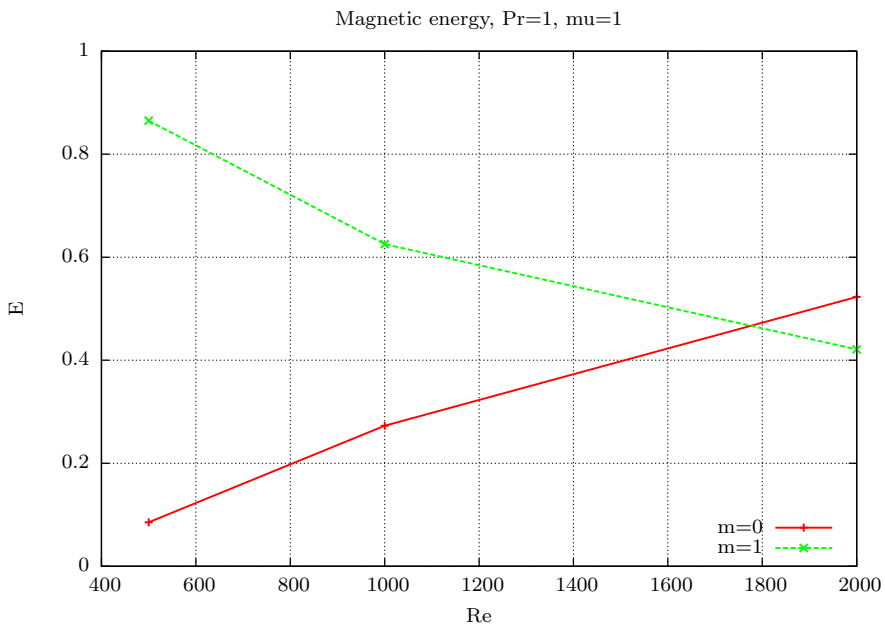




Permeability

# Mode and growth dependence on parameters

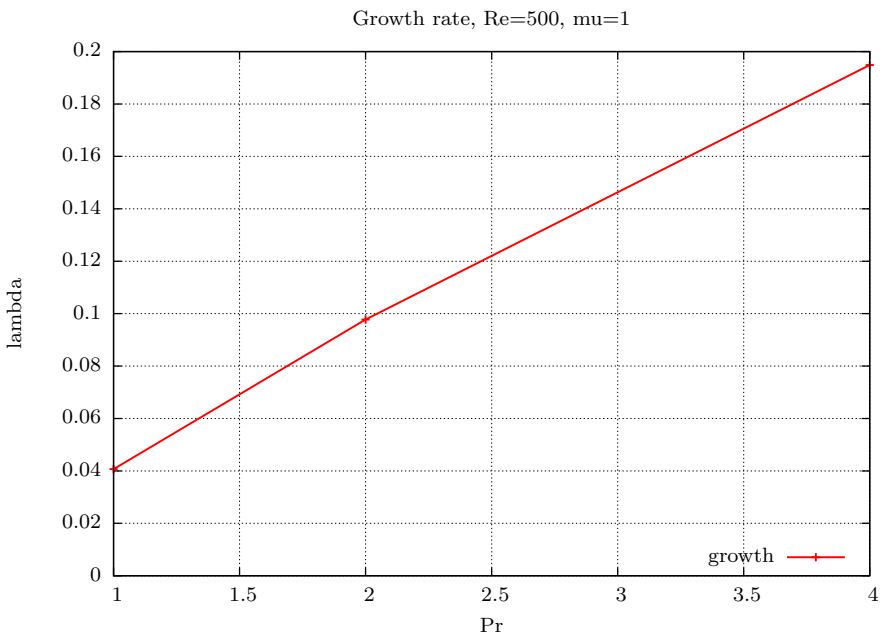
Preliminary results: Growth rate $\lambda$ $[T_{imp}]$ ($E_B \propto \exp(\lambda t)$ in kinematic phase) and magnetic modes $m$ ($B_m(r, \phi, z) = f_m(r, z)exp(im\phi)$) depend strongly on the system parameters $Re$, $Pr$, $\mu$.

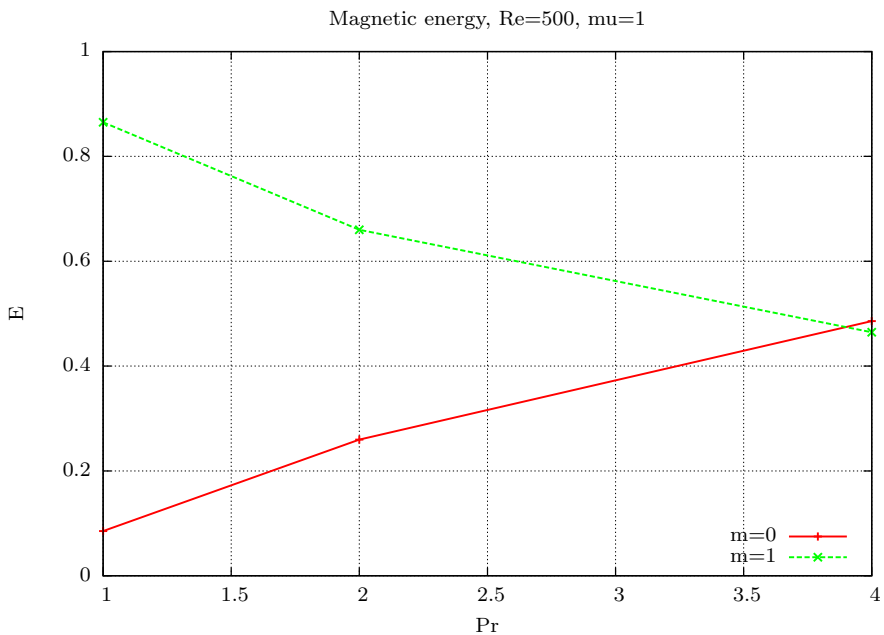


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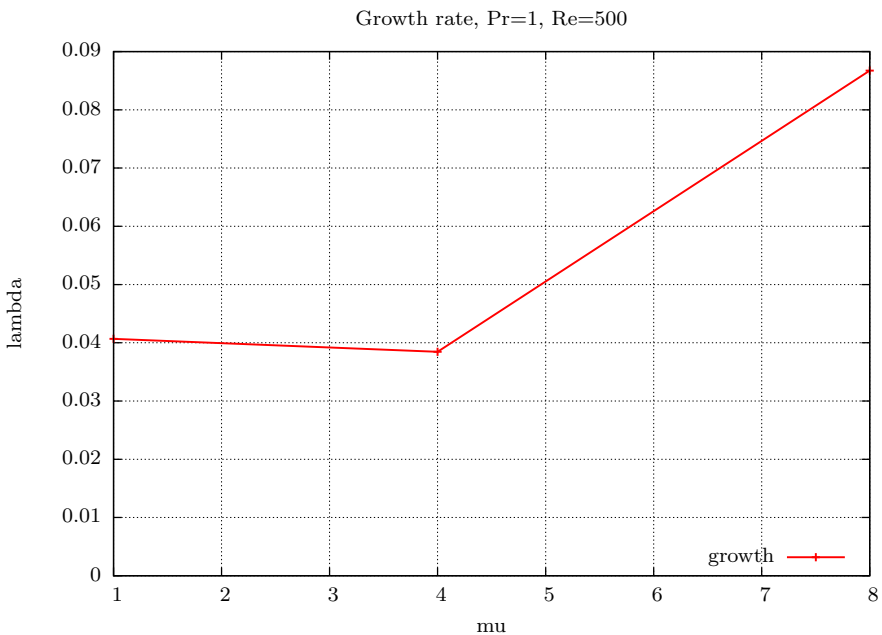
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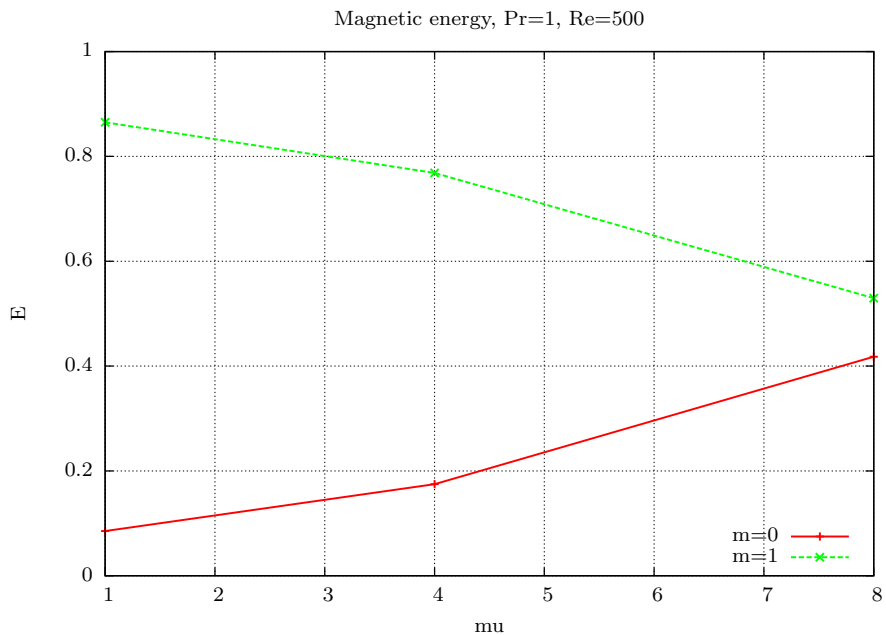


# Mode and growth dependence on parameters

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# Stability

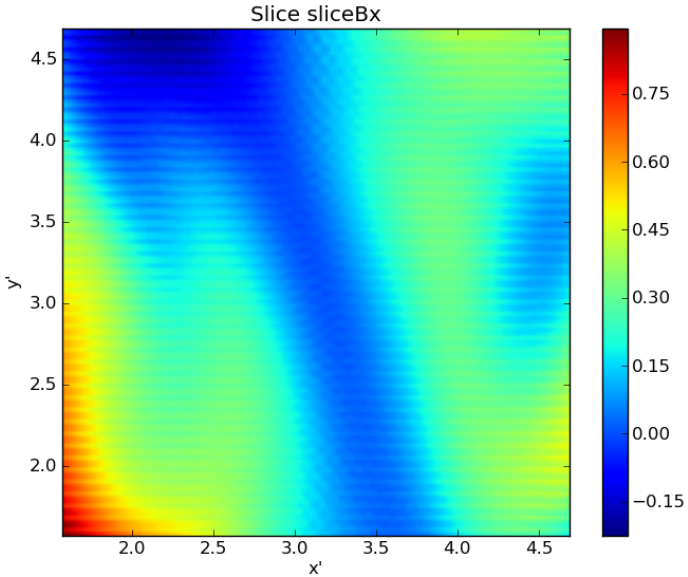
cfl condition limits possible material properties.

Typically only variation of $\mu$, since $\eta_{softiron} = \eta_{sodium}$.
The critical term is $rhs = ... - \eta \nabla \times \nabla \times \boldsymbol{B}/\mu$.

Fluid: $rhs = ... - \eta \nabla \times \nabla \times \boldsymbol{B} = ... + \eta \nabla^2 \boldsymbol{B}$, thus
$Rm_{max} = LU/\eta_{min}$
Solid: $rhs = ... - \eta/\mu \nabla \times \nabla \times \boldsymbol{B} = .. + \eta/\mu \nabla^2 \boldsymbol{B}$, thus
$Rm^{eff}_{max} = LU/(\eta/\mu)_{min}$

Permeability sets a restrictive limit for resistivity, when trying to keep stability.
But typical parameter regime in experiment: $Rm \approx 50$ (low),
$\mu \approx 40 \Rightarrow Rm^{eff} = 2000$

# Gibbs oscillations - smoothing




Problem: magnetic field is discontinuous due to jump conditions

Derivatives develop strong oscillations (stronger for higher permeability; Gibbs phenomenon)

Solution: smoothing via a sigma-filter; 1D: $u(x) = \sum_{k=-N/2}^{N/2} \sigma_k e^{ikx} \hat{u}_k$

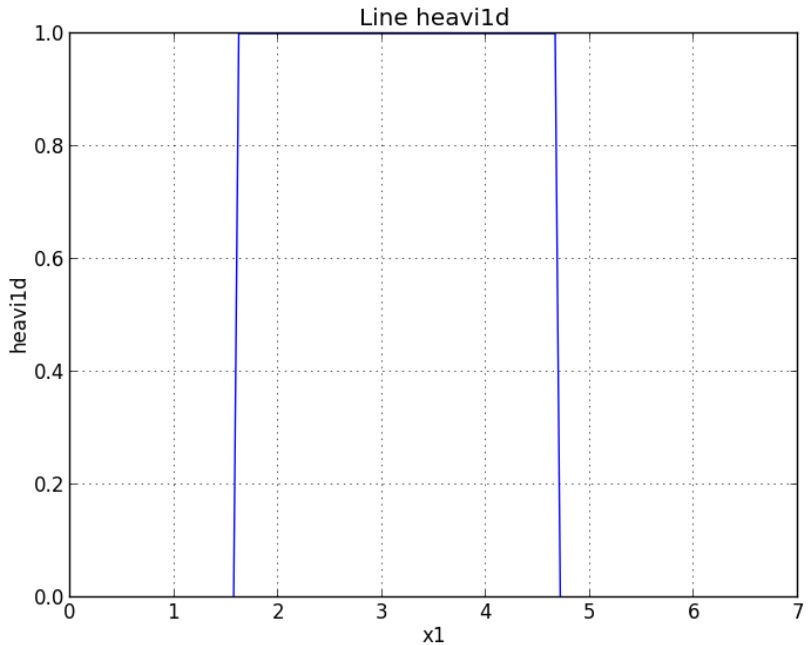
Cesaro: $\sigma_k = 1 - \frac{|k|}{N/2+1}$

Lanczos: $\sigma_k = \frac{\sin(2\pi k/N)}{2\pi k/N}$

Raised cosine: $\sigma_k^{(0)} = \frac{1+\cos(2\pi k/N)}{2}$

Sharpened raised cosine: $\sigma_k = {\sigma_k^{(0)}}^4 (35 - 84\sigma_k^{(0)} + 70{\sigma_k^{(0)}}^2 - 20{\sigma_k^{(0)}}^3)$

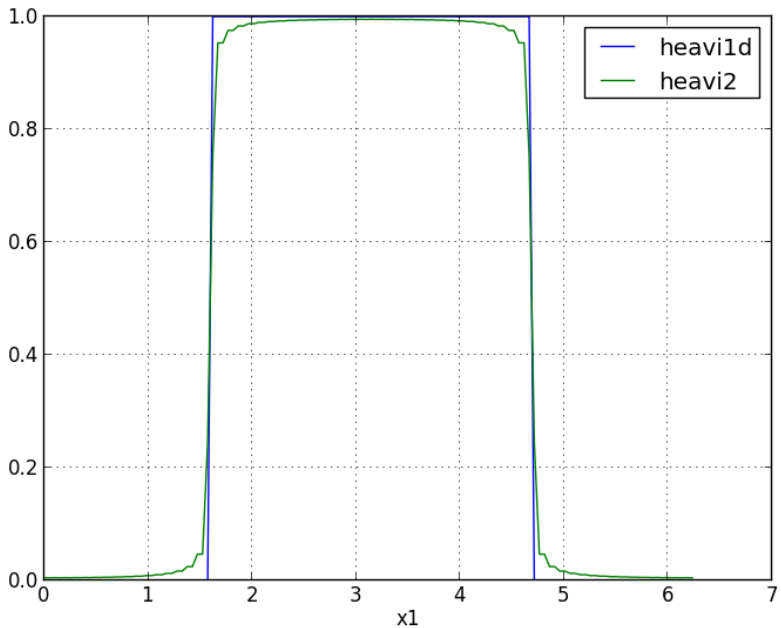
# Gibbs oscillations - smoothing

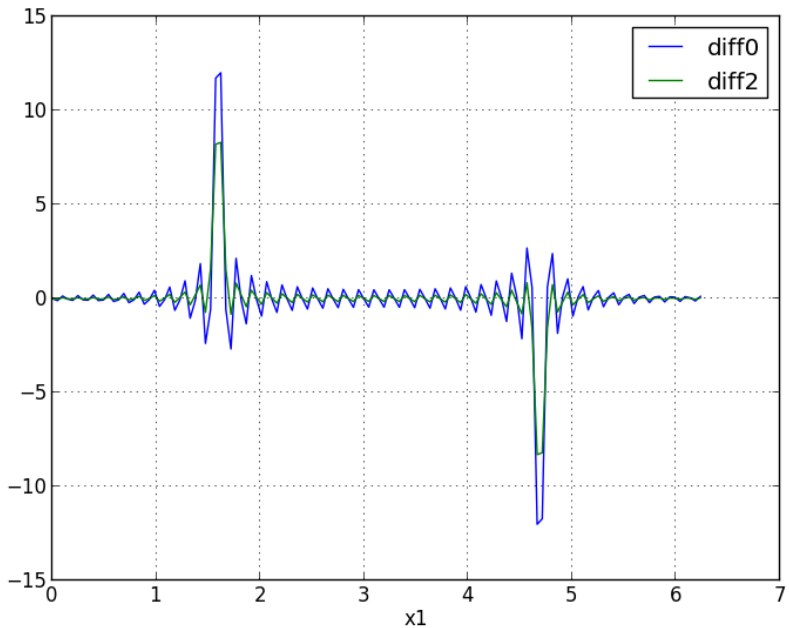



Toy problem: calculate the derivative of the periodic step function

Problem is similar to magnetic field jumps (since the permeability is a slightly smoothed step function) and shows the same problems

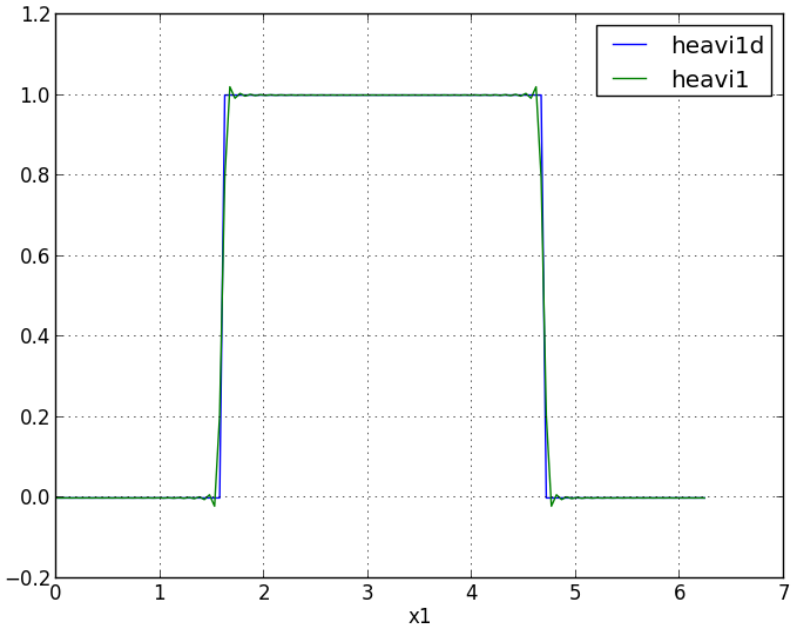
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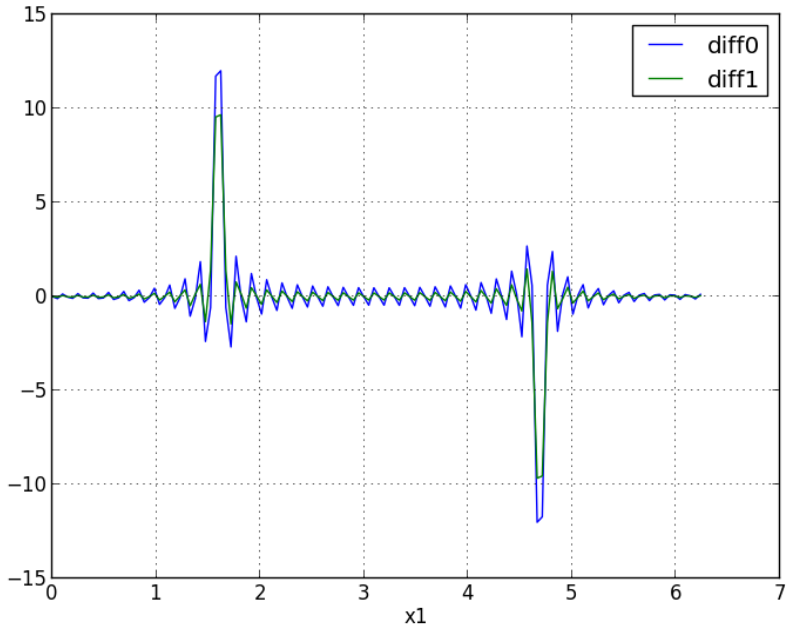



Cesaro smoothing

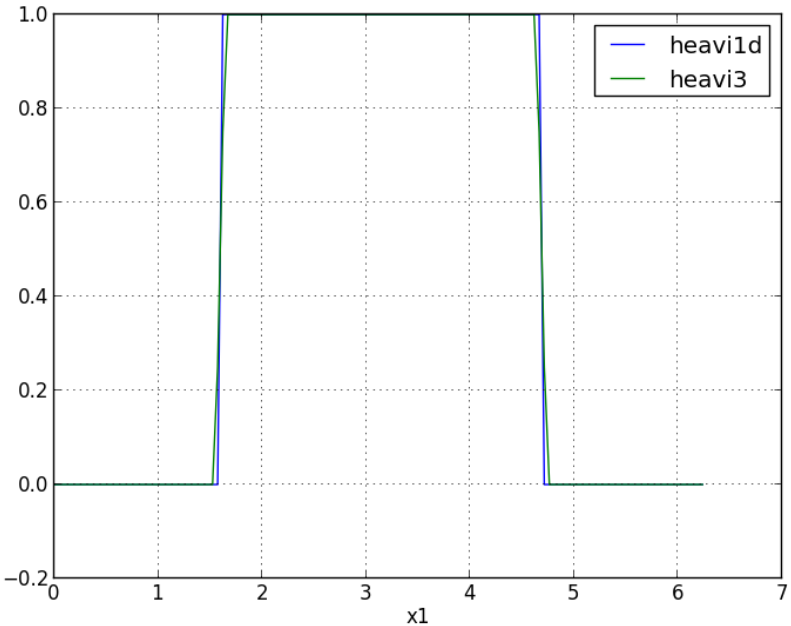
# Gibbs oscillations - smoothing

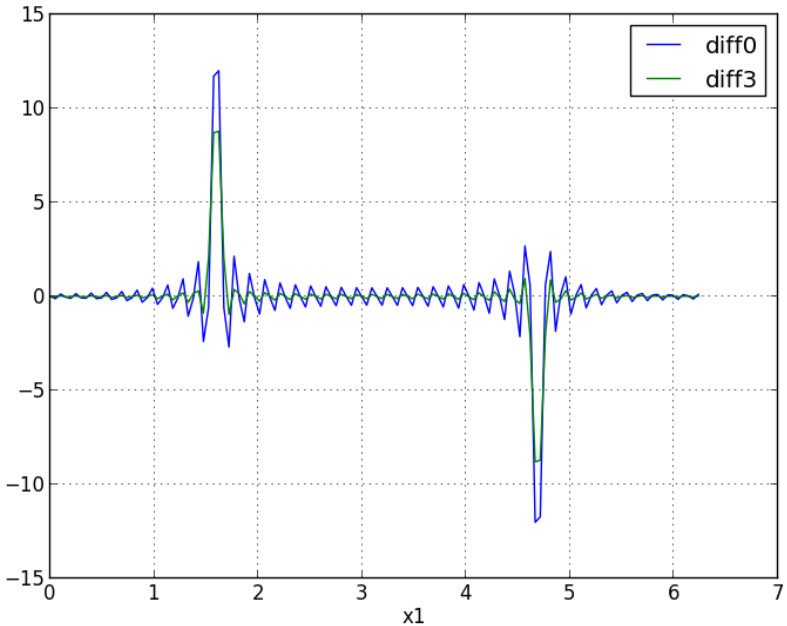



Lanczos smoothing

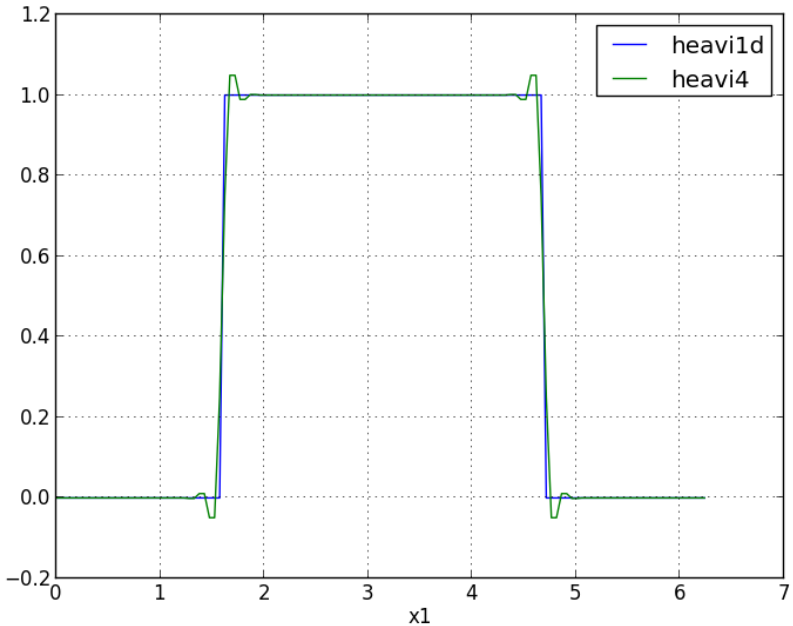
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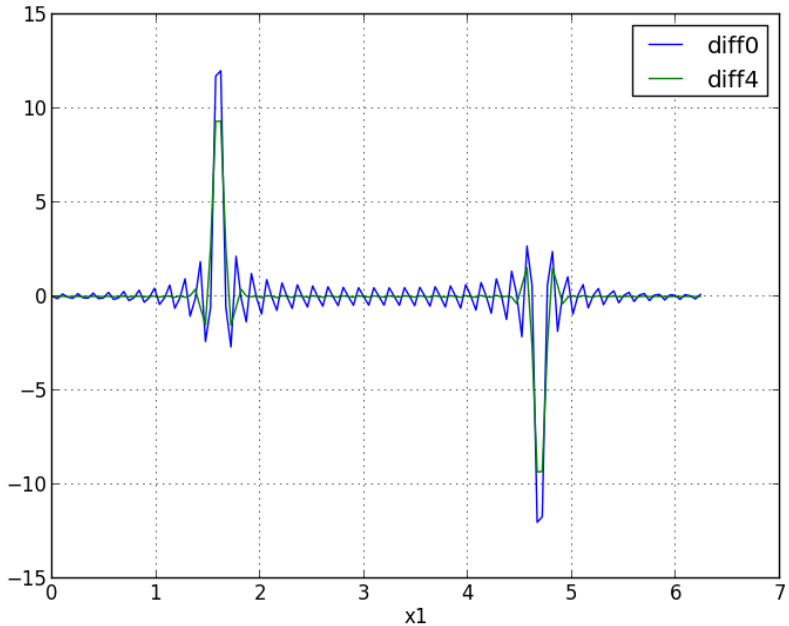



Raised cosine smoothing

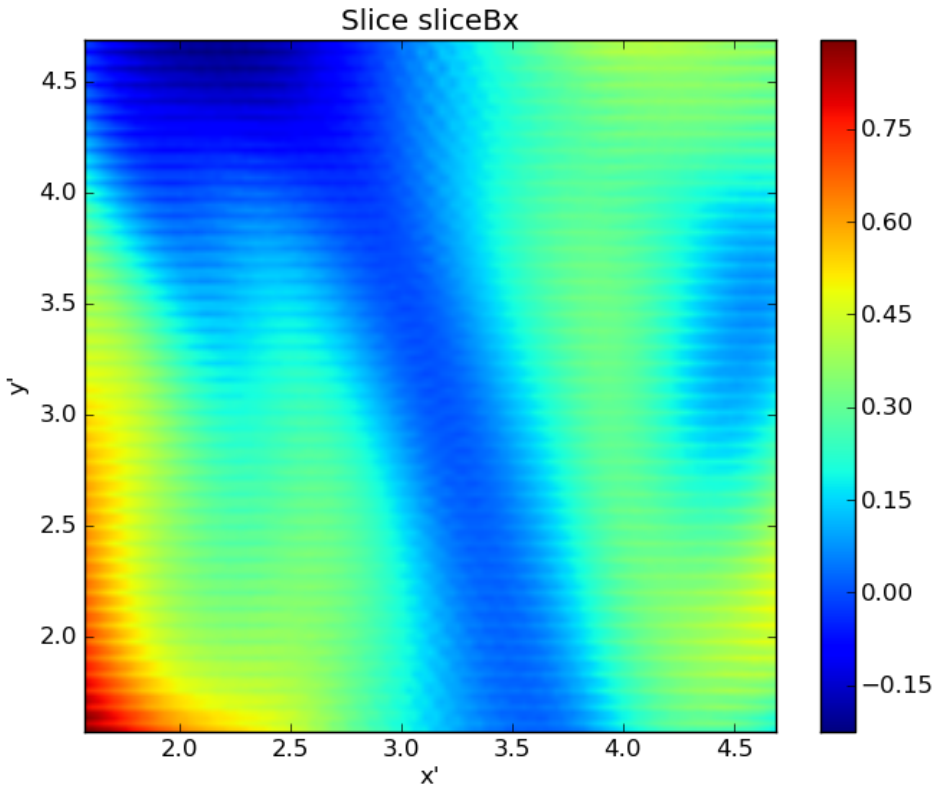
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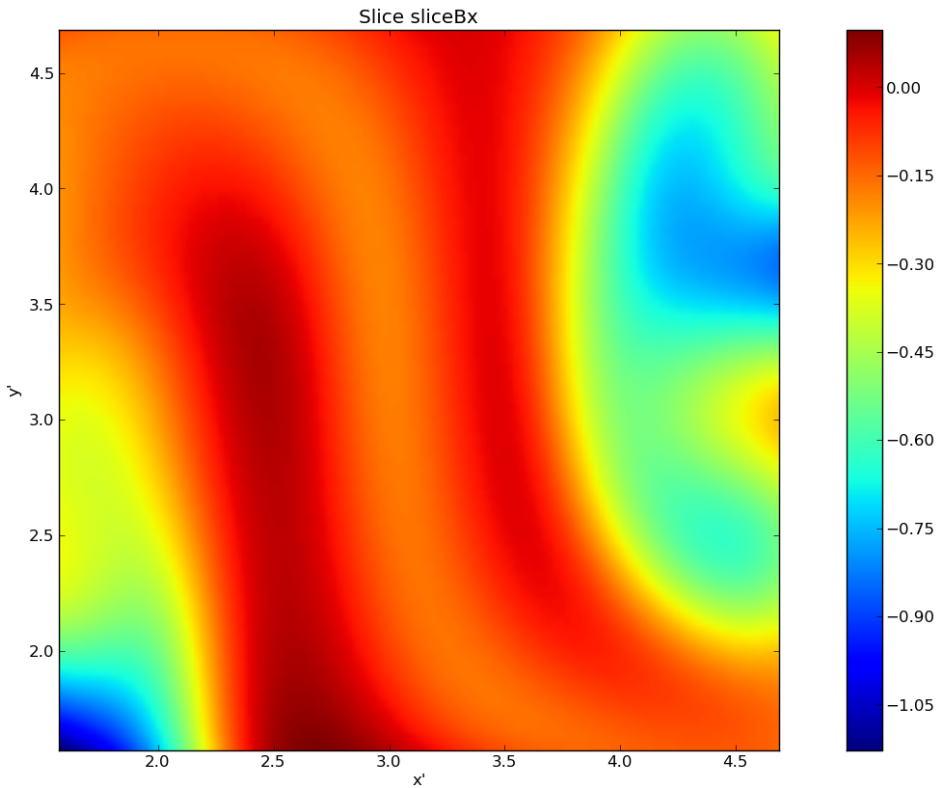



Sharpened raised cosine smoothing

# Gibbs oscillations - smoothing




Sharpened raised cosine implemented in LaTu

# Summary

- Rotation in Fourier space is a lot faster than in physical space, making higher order interpolation feasible.
- Outer boundary VTF condition can be treated with penalty, solid impellers are modeled using a modified induction equation.
- Benchmark against FV code shows qualitatively equal results: Conductivity enhances m=0 poloidal mode, permeability enhances m=0 toroidal mode.
- Growth rate increases with $Re$, $Pr$, $\mu$ as well as the m=0 mode, while m=1 mode decreases.
- Permeability is limited due to cfl condition.
- Gibbs oscillations can be reduced with a sigma-filter while maintaining reliable solutions.