

How to compute the light time for a planet?

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Abstract

This note explores and discusses the issues raised by the computation of the light time of moving bodies. It is shown that the classical approach stumbles quickly on the choice of the privileged reference frame for the light propagation and has difficulty to produce an undisputable result. Although the correction to the zero-oder term is in v/c, no satisfactory solution is found with Newtonian kinematics while a formulation in the framework of special relativity helps understand the problems and solve the conundrum. A fully covariant solution is provided which is correct to any order in v/c. The text addresses primarily the concepts in a didactic way rather than the details of an accurate modelling in order to emphasize on the critical features. As such it is a good starting point to stimulate further discussions.

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1 Introduction

To compute the apparent direction of a planet P observed by an observer located at O in the solar system, one needs to determine the light time of a photon between its emission from the planet at $t = t_e$ and its reception at $t = t_r$ by the observer. Basically the incoming photon comes from the direction where the planet was at the emission time and not where it lies at the reception time and this incoming direction is also the one where the moving bodies is seen at t_r. This effect is not applied to the stars since it has been an accepted practice in astronomy to consider the stellar positions in the barycentric frame from the apparent direction of the light. The difference between the geometric and apparent position due to the motion of the planets should not be confused with the so-called stellar aberration which results from the motion of the Earth (and more generally of the observer when a spacecraft is involved) in the barycentric frame and applies to both stars and planets.

If the distance of the planet to the observer is ρ, the light-time is about ρ/c and the apparent direction is close to the geometric position at the retarded time $t - \rho/c$. The difficulty arises when one asks the question about when this distance should be taken: at the emission time, reception time or something else? For a typical main-belt asteroid the light-time is $1000 - 2000$ s and the shift between the geometric position at t and $t - \rho/c$ is $\sim 10 - 20$ arcsec. For high precision astrometry this must be known to within few tens of μas, that is to say to one part in a million. This means that a relativistic formulation including terms of the order of v^2/c^2 is required. However the object of this note is more fundamental as it deals with the conceptual approach and aims to show that even to first order in v/c there is no satisfactory answer as to the best way to evaluate this effect in the Newtonian framework. A deeper analysis in special relativity helps identify the source of the difficulty and produces an unambiguous result.

2 Definition of the light time

Let the planet and the observer be referred to in a frame whose origin is at the barycentre of the solar system (Fig. 1). The position vector of the planet is $\mathbf{BP}(t)$ and that of the observer $\mathbf{BO}(t)$, or equivalently $\mathbf{r}_p(t)$ and $\mathbf{r}_o(t)$. The light time equation at the observing time t is given by

$$c\,\Delta t = |\mathbf{BP}(t - \Delta\, t) - \mathbf{BO}(t)| \tag{1}$$

where c is the light velocity. Equation 1 can be solved to full precision by iterations or with a sufficient approximation for solar system observation in a close form by expanding the first term in the right-hand side to first order in $\Delta\, t$ as,

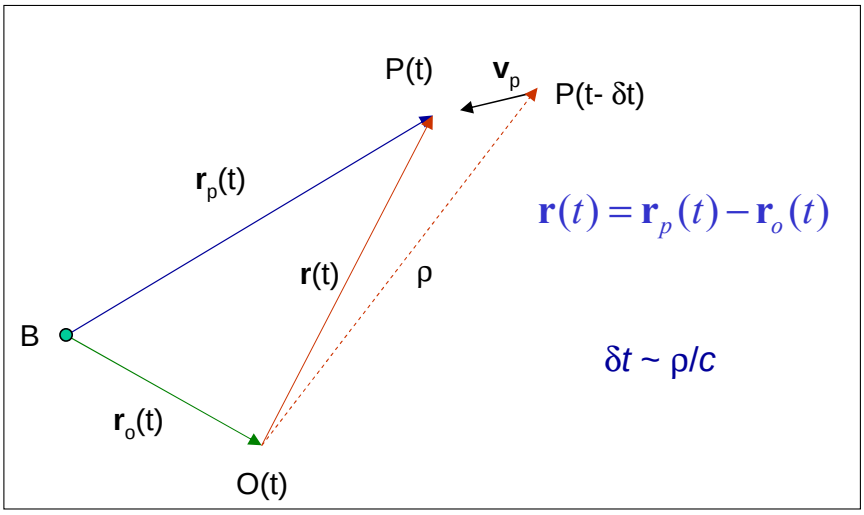


FIGURE 1: Position of the planet and the observer in the barycentric frame

$$\mathbf{BP}(t-\Delta t) \approx \mathbf{BP}(t) - \mathbf{V}_p(t)\,\Delta t \tag{2}$$

or

$$c\Delta t = |\mathbf{OP}(t) - \mathbf{V}_p(t)\,\Delta t| \tag{3}$$

which yields at the same level of approximation,

$$c\Delta t = r(t) - \frac{\mathbf{V}_p(t)\cdot\,\mathbf{OP}(t)}{r(t)}\Delta t \tag{4}$$

where $r(t) = |\mathbf{OP}(t)|$. By introducing the unit vector $\mathbf{u}$ in the direction $\mathbf{OP}(t)$ this is equivalent to,

$$c\Delta t = r(t) - \mathbf{V}_p(t)\cdot\,\mathbf{u}\,\Delta t \tag{5}$$

and finally,

$$\boxed{\Delta t = \frac{r(t)}{c}\,\frac{1}{1+\dfrac{\mathbf{V}_p(t)\cdot\,\mathbf{u}}{c}}} \tag{6}$$

In Eq. 6 the correcting term $\mathbf{V}_p(t) \cdot \mathbf{u}$ is the contribution of the barycentric motion of the planet to the radial velocity and should not be confused with v_r the relative radial velocity (usually referred to as the radial velocity). The above derivation is based on the assumption that the motion of the planet is quasi-uniform over the time scale comparable to the light-time, so that the acceleration effect can be neglected.

The derivation of the light time has been done in the barycentric frame, but the origin of the frame is in principle irrelevant and the observer could have found much simpler to work out the same expression in another frame. By selecting another frame in motion with respect to the barycentric frame, the only change that will show up in (6) will be in the planet velocity $\mathbf{V}_p$. This means that the light-time which seems to be a real observable, is not invariant with the frame selection, as if there were a privileged frame where it should be computed.

In particular it is natural for the observer to work out the light time in his own frame by using the relative motion $\mathbf{OP}(t)$. The initial equation (1) would have been,

$$c\,\Delta t = |\mathbf{OP}(t - \Delta t)| \tag{7}$$

Since $\mathbf{OP}(t - \Delta\, t) = \mathbf{BP}(t - \Delta\, t) - \mathbf{BO}(t - \Delta\, t)$, Eqs. 1 and 7 are not compatible.

The same derivation as before would lead to ,

$$\boxed{\Delta\, t = \frac{r(t)}{c}\,\frac{1}{1 + \dfrac{v_r}{c}}} \tag{8}$$

where $v_r = dr/dt$ is the full radial velocity including the contribution of the barycentric motion of the planet and that of the observer. Clearly the light times evaluated by (6) or (8) would not be identical. Likewise the origin at B in Eq. 1 could be changed to another point B' in motion with respect to B, so that the velocity of the planet in Eq. 6 would be different yielding another value for $\Delta\, t$. Nothing is totally wrong there, but at least this calls for a more careful definition of the light time to get an unambiguous value which at the minimum is independent of the reference frame. It is perfectly conceivable that two independent astronomers end up one with (6) and the other with (8), each being certain to have made a flawless and accurate derivation. At the end they won't get identical numbers in their software and the cause will be hard to trace since the differences in apparent positions will be of the order of 0.1 to 1 mas. These numbers are small for traditional solar-system astrometry and this is probably the reason why, at least to the author knowledge, no one took the pain to look at the issue in depth.

Since the above derivations are purely Newtonian, it can be argued that the velocity of light is not the same in both frames and the consistency could be obtained by composing the velocity

of light with that of the observer and this is largely correct. But this raises difficult questions about the frame in which this velocity must be taken equal to the standard c value, very akin to the discussion at the end of the 19th century which led to the Michelson-Morley experiment. If the result is correct in the barycentric frame, how should it be corrected if the observer and the planets were referred to a non-rotating frame with origin at the galactic center? Should we add the translational velocity of the solar system around the galactic center to the velocity of light? In which of these frame the defining value of c should be used? It is clear that there is a single answer to the question of the magnitude of the light-time and neither of the above formulas gives us an undisputable value, although both are nearly correct. Taking the velocity of light unchanged from frame to frame is clearly wrong in Newtonian kinematics, but while this clarifies somewhat the problem this is not of great help for the practical computation.

Using an invariant velocity of light in Newtonian kinematics would amount to accepting the Einstein invariance of the speed of light, without using its kinematical consequences to deal with space time events, a conspicuous inconsistency. Hence one may expect that a special relativity treatment will be unambiguous and this is true. However this is not fully satisfactory at this stage since we are concerned with v/c effects and a classical derivation should be sufficient without invoking the subtleties of special relativity.

3 A simplified model

In order to grasp the essence of the problem and to avoid to have it hidden behind unnecessary formalism I restrict now to a simplified version in one dimension. Both the observer and the planet (the light source) are moving on a straight line at uniform velocity as sketched out in Fig. 2.

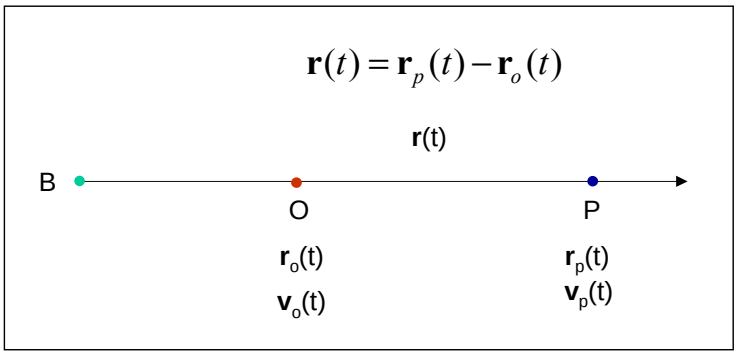


FIGURE 2: Position of the planet and the observer in the 1-D model

3.1 Newtonian derivation

We assume now that the velocity of light is c (whatever its value) in the barycentric frame when emitted by a source at rest in this frame. There are two sub-cases according as there is a medium where the light propagates, so that its velocity does not depend of the motion of the source from which the light signal is emitted, or whether there is no medium and the light signal has a c speed in the frame of the source. We will show now that a Newtonian treatment with galilean composition of the velocities provides a full consistency. However this does not tell us what value of the light velocity should be used in the numerical evaluation.

3.1.1 Analysis in the fixed frame

In the first case one has for the light-time equation,

$$r_p(t_e) - r_o(t_r) = c(t_r - t_e) = c\Delta\, t \tag{9}$$

or with $t_r = t$

$$r_p(t - \Delta\, t) - r_o(t) = c\Delta\, t \tag{10}$$

which gives (exactly for a uniform motion),

$$\Delta\, t = \frac{r(t)}{c}\,\frac{1}{1+\dfrac{v_p}{c}} \approx \frac{r(t)}{c}\left(1 - \frac{v_p}{c}\right) \tag{11}$$

where $r(t)$ is the distance between the observer and the planet when the light reaches the observer.

3.1.2 Analysis in the observer frame

The light velocity is now $c + v_o$ and the light time equation reads,

$$r(t_e) = (c + v_o)(t_r - t_e) \tag{12}$$

but $r(t_e) = r(t_r - \Delta\, t) = r(t_r) - v_r\, \Delta\, t$ where v_r is the radial velocity of the planet with respect to the observer. This gives to the first order in v/c

$$\Delta\, t \approx \frac{r(t)}{c}\left(1 - \frac{v_o + v_r}{c}\right) = \frac{r(t)}{c}\left(1 - \frac{v_p}{c}\right) \tag{13}$$

since $v_r = v_p - v_o$ in Newtonian kinematics.

Now equations 11-13 are un full agreement.

Had we considered that the velocity of light is c in the frame of the source emitting the light, the derivation would not have changed very much. The only change is then to replace the velocity of light in the fixed frame by $\tilde{c} = c - v_p$. Using this $\tilde{c}$ in Eqs. 11-13 one would get,

$$\Delta t \approx \frac{r(t)}{c} \tag{14}$$

This is a different result from Eq. 11, but it may be made consistent if we consider that the velocities of light are not taken in the same frame and have not the same numerical value. However for the practical astronomer it is hard to decide which should be used for an accurate computation of the light time, with significant consequences for the evaluation of the planetary aberration, even at the first order in v/c. In addition whatever the expression used there is good chance that in either case he takes $c = 299,792,458$ km s^{-1}.

The above derivations raise serious questions about the very nature of the light propagation and this is one of the rare instance where a Newtonian derivation of a simple phenomena to the first order in v/c cannot be safely handled. The who wishes to understand the background instead of applying blindly a formula taken in a textbook is immediately confronted to fundamental questions that have haunted physicists for decades. Although the discussions by Lorentz or Einstein on the issue of the propagation of light in different frames were not done in this framework, their concerns were totally equivalent and the debates around the stellar aberration (very similar to the light time problem) were central in the downfall of the aether theory. A student faced with the above derivations will be quickly made sensitive to the issue of the absolute space and I think this example could be a nice tool in the classroom.

3.2 Relativistic derivation: first order

Special Relativity provides us with a more secure basis to carry out this derivation, but it remains puzzling that it is required at this level of accuracy, since no terms in v^2/c^2 are involved. In the following I call the barycentric frame the *fixed frame* with coordinate x, t and the observer frame is the *moving frame* with spacetime coordinates X, T.

3.2.1 Analysis in the fixed frame

The notations are slightly changed to better agree with the standard expressions of the Lorentz transformation. They are shown in Fig. 3 drawn in the fixed frame.

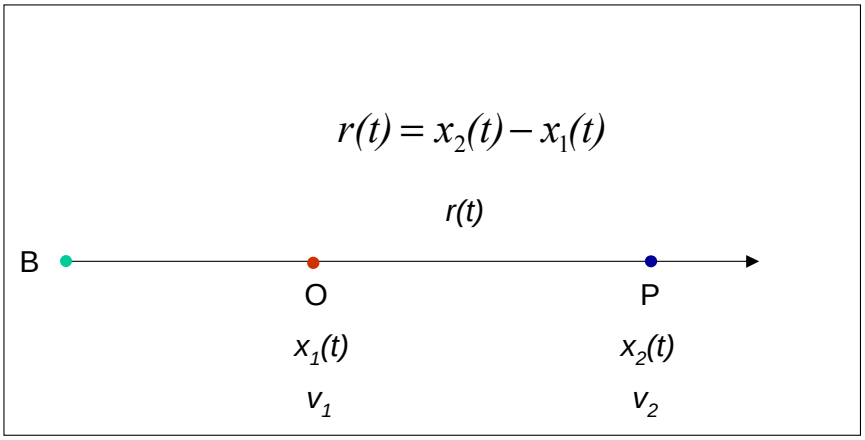


FIGURE 3: Position of the planet and the observer in the 1-D model with standard notations

The light-time equation becomes,

$$x_1(t_r) + c(t_r - t_e) = x_2(t_e) \tag{15}$$

which for uniform motions leads to,

$$\Delta t = \frac{r(t)}{c} \frac{1}{\left(1 + \frac{v_2}{c}\right)} \approx \frac{r(t)}{c} \left(1 - \frac{v_2}{c}\right) \tag{16}$$

3.2.2 Analysis in the moving frame

By applying the same analysis in his own frame the observer concludes that the light-time is given by,

$$\Delta T = \frac{R(T)}{c} \frac{1}{\left(1 + \frac{v_r}{c}\right)} \approx \frac{R(T)}{c} \left(1 - \frac{v_r}{c}\right) \tag{17}$$

where v_r is the velocity of the planet in the observer frame, $R(T)$ is the position of the planet at the time T in the same frame at the arrival time of the light signal.

Eqs. 16 and 17 are the equivalent of (11)-(13) but with the coordinate times of each frame. The comparison between Eqs. 16 and 17 is quite interesting since, as expected from the previous section, to the first order in v/c they disagree, as $v_r \approx v_2 - v_1$ and $R(T) \approx r(t)$. So it seems that applying a Lorentz transformation for T and v_r won't solve the problem, since the

difference between the Galilean transformation and the Lorentz transformation will be made only of second order terms. One should note also that the expressions are rigourous for a uniform motion and that, unlike with the Newtonian derivation, the velocity of light is the same in both formulas. As a consequence the difference of the order of v_1/c remains really puzzling and because of the invariance of the velocity of light one seems farther from the consistency as we were with the Newtonian model.

We prove however now that the two expressions are related by a Lorentz transformation and either of the two can be considered as a Lorentz invariant form of the light-time.

$$X = (x - v_1\, t)\, \gamma \tag{18}$$
$$T = \left(t - \frac{v_1\, x}{c^2}\right) \gamma \tag{19}$$

$$x = (X + v_1\, T)\, \gamma \tag{20}$$
$$t = \left(T + \frac{v_1\, X}{c^2}\right) \gamma \tag{21}$$

where as usual,

$$\gamma = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2} \tag{22}$$

One has also the relative velocity v_r as,

$$v_r = \frac{v_2 - v_1}{1 - \frac{v_2\, v_1}{c^2}} \tag{23}$$

The key element of the proof is the fact that the light times Δt or ΔT are the differences of time between two space-times events and neither is a proper time of the observer.

Let t_e and t_r be the emission and reception time of the light signal in the fixed frame. They are associated to two spacetime events of coordinates $(x_e = x_2(t_e), t_e)$ and $(x_r = x_1(t_r), t_r)$. In the moving frame the coordinates of the **same** events are,

$$X_e = (x_e - v_1\, t_e)\,\gamma \tag{24}$$
$$T_e = \left(t_e - \frac{v_1\, x_e}{c^2}\right)\,\gamma \tag{25}$$

and

$$X_r = (x_r - v_1\, t_r)\,\gamma \tag{26}$$
$$T_r = \left(t_r - \frac{v_1\, x_r}{c^2}\right)\,\gamma \tag{27}$$

By combining the two transformations one has,

$$\Delta T = T_r - T_e = \gamma\left((t_r - t_e) - \frac{v_1}{c^2}(x_r - x_e)\right) \tag{28}$$

and with Eq. 15 one gets,

$$\boxed{\Delta T = T_r - T_e = \gamma\,\Delta\, t\left(1 + \frac{v_1}{c}\right) \approx \Delta\, t\left(1 + \frac{v_1}{c}\right)} \tag{29}$$

which leads to an unexpected first order effect linked to the relativity of simultaneity in light signals exchanged between two points. This is the exact counterpart of our using the invariance of the velocity of light in Eqs. 16- 17. At least we have proved that the difference between $\Delta\, t$ and $\Delta\, T$ in (16)-(17) is explained by a Lorentz transformation limited to the first order in v/c. (I show below that they indeed agree to all orders in v/c).

However this is not the end of the story as either derivation could have been used to first order in v/c to compute the light time with $\Delta\, t$ or $\Delta\, T$, with clearly two different values and the above derivation does not provide guidelines as to the correct way of doing the computation. In the moving frame Eq. 17 applies and gives the light time in the coordinate time of the moving frame. It can be expressed in the coordinate time of the fixed frame with (29) and the results would agree with a direct derivation performed in the fixed frame. Nevertheless it is unusual at this level of approximation (v/c) that one has to take care of effects related to the relativity of simultaneity as soon as we assume the invariance of the speed of light. Further investigation is clearly required here for a better understanding that would make the mathematical result more acceptable.

I have used repeatedly the argument that the correction to the zero-order term (light time expressed as r/c) is in first order in v/c and got very surprised by the fact this can only be properly explained by using special relativity concepts as relativity of time or that of simultaneity. But this idea of first order terms is disputable since the actual correction in the position of the moving body is typically,

$$\frac{v_p\,\Delta t}{r} \approx \frac{v_p}{c} \tag{30}$$

meaning that what I called the zero-oder term is in fact comparable to the usual stellar aberration in v/c. So the correction term for Δt in v_r/c will show up in the apparent position as a second order term in v/c, like the second order aberration which is purely relativistic. From that viewpoint it is easier to understand why the Newtonian kinematics runs into trouble to model the phenomena. This makes the topic even more valuable to introduce a discussion of the speed of light and the absolute space, since the Newtonian modelling is immediately paradoxical.

3.3 Relativistic derivation: covariant expression

Let us show now that Eqs. 16 is in fact the covariant expression of the light time in the framework of special relativity, meaning that it is correct in any inertial frame without approximation. First, one can already say that the derivation of

$$\Delta\, t = \frac{r(t)}{c}\frac{1}{\left(1+\frac{v_2}{c}\right)} \tag{31}$$

has been done without approximations going beyond the model of rectilinear uniform motion. Therefore the expression is quite general and depends only the velocity of the source of light and on the distance between the source and the observer at the reception time. Assuming it is expressed in a covariant form, it is valid in any frame and in particular in the moving frame. In this case the equivalent expression must be precisely Eq. 17

$$\Delta\, T = \frac{R(T)}{c}\frac{1}{\left(1+\frac{v_r}{c}\right)} \tag{32}$$

where the velocity of the source is replaced by the relative velocity of the planet with respect to the observer. The distance of the light source at the reception time is also replaced by a similar quantity expressed in the moving frame. The transformation between $r(t)$ and $R(T)$ is not trivial and detailed below.

The object of this section is to prove that Eqs. 31 and 32 are related by a Lorentz transformation and express the same physical quantity as seen from two reference frames in relative uniform motion.

We have already found the relationship between Δt and ΔT in Eq. 29 with

$$\boxed{\Delta T = T_r - T_e = \gamma \, \Delta t \left(1 + \frac{v_1}{c}\right)} \tag{33}$$

Come now to the trickiest point by relating $r(t)$ and $R(T)$. At first glance it seems that they should be related like a measurement of the length of a rod carried out in the two frames. But while both can be seen as length measurement in their own frame (position of the ends of a stick at the same time in either frame), this is not the same stick which is considered in the fixed and moving frames. $R(T)$ is in fact $X_2(T_r)$, the coordinate of the light source in the moving frame when the light reaches the observer, and $r(t)$ is $x_2(t_r) - x_1(t_r)$ the difference of abscissas between the light source and the observer at reception time as seen in the fixed frame.

In the fixed frame the motions of the observer and the planet are given by,

$$x_1(t) = v_1 \, t \tag{34}$$
$$x_2(t) = a + v_2 \, t \tag{35}$$

where the origin of time is so selected that the origin of the moving frame coincides with that of the fixed frame at $t = 0$. Then at $X = 0$ one has also $T = 0$ when the two origins coincide. At $t = 0$ one has $r(0) = a$, the position of the planet at the origin of time. At any time t one has then,

$$r(t) = a + (v_2 - v_1) \, t \tag{36}$$

In the moving frame the motions are described by,

$$X_1(T) = 0, \quad \forall \, T \tag{37}$$
$$X_2(T) = b + v_r \, T \tag{38}$$

where v_r is the relative velocity of the light source with respect to the observer in the moving frame. It is given by Eq. 23 in terms of v_1 and v_2. At any time one has $R(T) = X_2(T)$ and its initial value at $T = 0$ is b that is still unrelated to a.

At the origin of time when the two reference frames coincide, the fixed observer says the planet is at a distance a from the origin while the moving observer sees it at a distance b from its origin. In both cases they compare the position of the planet to their position at $t = 0$ for the fixed observer and at $T = 0$ for the moving observer. Therefore the positions of the planet are not associated to a single and well defined space time event, because outside the origin of coordinates what is simultaneous from one observer, is not for the other. There are in fact two distinct space time events: $E_1 : (b, T = 0)$ in the moving frame and $E_2 : (a, t = 0)$ in the fixed frame that determine the planet position for each observer at the origin of time.

Starting from E_1 its coordinates in the fixed frame are $E_1 : (b\,\gamma,\ \frac{v_1\,b}{c^2}\,\gamma)$. Therefore for the fixed observer the event E_1 does not take place at $t = 0$ and for him the planet has moved over a duration $\delta\,t = \frac{v_1\,b}{c^2}\,\gamma$ in comparison to the position he assigned to it at his origin of time. Hence its location is given by,

$$x_2(\delta\,t) = a + v_2\,\frac{v_1\,b}{c^2}\,\gamma \tag{39}$$

and this is equal to $b\gamma$, the location of E_1 in the fixed frame coordinates.

This allows to relate a and b with,

$$\boxed{a = b\,\gamma\left(1 - \frac{v_1\,v_2}{c^2}\right)} \tag{40}$$

The same reasoning could have been done with the other perspective by starting from E_2. Its coordinates in the moving frame are $(a\,\gamma, -\frac{v_1\,a}{c^2}\,\gamma)$ and for the moving observer the planet has not yet reached its position at $T = 0$. It must move over a duration $\delta\,T = \frac{v_1\,a}{c^2}\,\gamma$ and then its location becomes in the moving frame,

$$X_2(\delta\,T) = b - v_r\,\frac{v_1\,a}{c^2}\,\gamma \tag{41}$$

This must be equal to $a\gamma$ yielding,

$$\boxed{b = a\,\gamma\left(1 + \frac{v_1\,v_r}{c^2}\right)} \tag{42}$$

Using Eq. 23 for v_r it is easily shown that Eqs. 40 and 42 are identical. This gives also the useful identity,

$$\gamma^2 \left(1 + \frac{v_1\, v_r}{c^2}\right) \left(1 - \frac{v_1\, v_2}{c^2}\right) = 1 \tag{43}$$

One considers now the space time event 'arrival of the photon at the observer', which is a well defined event whose coordinates can be computed in both frame as $t_r, x_1(t_r)$ in the fixed frame and $T_r, X_1 = 0$ in the moving frame. Therefore one has with Eq. 21

$$t_r = \gamma\, T_r \tag{44}$$

which says nothing more than T_r is the duration between the origin of time and the catch of the photon by the observer expressed in the proper time of the observer, that is to say read on his own clock.

At any time T in the moving frame one has for an event (X_2, T),

$$R(T) = X_2(T) = b + V_r\, T \tag{45}$$

or equivalently with (44) and (40),

$$R(T) = X_2(T) = \frac{a}{\gamma \left(1 - \frac{v_1\, v_2}{c^2}\right)} + \frac{v_2 - v_1}{1 - \frac{v_1\, v_2}{c^2}} \frac{t}{\gamma} \tag{46}$$

and by factoring,

$$R(T) = X_2(T) = \frac{1}{\gamma \left(1 - \frac{v_1\, v_2}{c^2}\right)} \left[a + (v_2 - v_1)\, t\right] \tag{47}$$

which finally gives with (36)

$$\boxed{R(T) = \frac{1}{\gamma \left(1 - \frac{v_1\, v_2}{c^2}\right)} r(t)} \tag{48}$$

Remark: The Lorentz transformation between $R(T)$ and $r(t)$ is not similar to a length contraction. This was expected since, although the quantities are very close to rod length, they do not not correspond to the measure of a particular length by two observers.

We have now all in hands to complete the proof that (31) and (32) are related by a Lorentz transformation.

From (33) and (32)

$$\Delta t = \frac{\Delta T}{\gamma \left(1 + \frac{v_1}{c}\right)} = \frac{R(T)/c}{\gamma \left(1 + \frac{v_1}{c}\right) \left(1 + \frac{v_r}{c}\right)} \tag{49}$$

and with (48)

$$\Delta t = \frac{r(t)/c}{\gamma^2 \left(1 + \frac{v_1}{c}\right) \left(1 + \frac{v_r}{c}\right) \left(1 - \frac{v_1\, v_2}{c^2}\right)} \tag{50}$$

which after little algebra gives,

$$\boxed{\Delta\, t = \frac{r(t)}{c} \frac{1}{\left(1 + \frac{v_2}{c}\right)}} \tag{51}$$

identical to (31). This establishes that this expression is the covariant form of the light time applicable in any inertial frame.

4 Conclusion

In this note I have attempted to shed light on the rigourous way to compute the light time between a moving body and an observer. While the relativistic approach has allowed to understand why the Newtonian results look paradoxical and how the different derivations can be reconciled, a more insightful approach is still missing. However I have shown that a fully relativistic (in the framework of special relativity) covariant expression can be derived which in principle is safe for the user.