Considerations on the orbit of GAIA for simulations

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15 March, 2002

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GAIA_FM_011

Abstract

Useful properties of the location of L2 and of the quasi-periodic orbit about L2 are derived from the linearized equations of motions. A practical orbit of GAIA around L2 avoiding the central eclipses (all annular) is provided, that meets the needs of the simulation.

1 Introduction

In the current baseline of the GAIA mission an orbit around the lagrangian point L2 has been selected as the best compromise between contradictory requirements, such as the radiocommunication, the telemetry rate, the thermal stability of the environment, the absence of eclipses. As is well known in celestial mechanics, the three equilibrium points aligned with the two primaries (here the Sun and the Earth) are not stable against small motions. A randomly selected initial motion about any of these points leads to an exponential drift, with a characteristic timescale of about three to six months. However it is known that well chosen initial conditions allow a point mass to wander about the lagrangian points on quasi-periodic orbits (referred to below as *Lissajous orbits*[1]) and even in very specific cases, on periodic orbits (known as *halo orbits*). These orbits are unstable, but can be easily controlled with cheap manoeuvres involving weak thrusts.

[1]Named after the French physicist Jules Antoine Lissajous (1822-1880) who investigated beats with vibrating mirrors mounted on two turning forks vibrating at right angle. The curves are also known as Bowditch curves from the American translator and commentator of the masterpiece of Laplace, *La Mécanique Céleste*, a work which marked the beginning of American participation in the field of celestial mechanics.

While the departure of the observing platform from the centre of the Earth is not a serious problem in the simulation of stellar positions, it must however be introduced at an early level in the simulations of solar system objects because of the significant excursion of the spacecraft around L2 compared to the size of the Solar System. For the stars, the additional and variable velocity of ~ 0.1 km s $^{-1}$ in the motion relative to L2 must also be included to compute the aberration effect. Finally the precise direction of the Sun as seen from the spacecraft is needed to assess properly the effect of the light deflection. For all these reasons it is important to include a rather realistic spacecraft motion in simulations.

In this note I summarize the properties of the orbit selected for GAIA in terms of position and velocity vectors that could be used in simulation work for GAIA. Fortran subroutines including the motion of the Earth with respect to the barycentre of the solar system and the motion of the Earth referred to the J2000 ecliptic and equinox will be made available shortly.

Most of the results presented in this note are classical in celestial mechanics, although several developments more specific to L1 and L2 are not found in textbooks. The quasiperiodic orbit about L2 has been investigated for GAIA and FIRST by M. Hechler [1] of ESOC. However this report is much more concerned by the problem of going to L2 with an Ariane launcher economically rather than with the detailed investigation of the small motion about L2. As the eclipse avoidance is critically related to the precise frequencies of the small motion about L2, I considered important to look thoroughly into the problem of the Lissajous orbit(in the linear approximation) around L2, having in view to supply a solution for the simulations easy and cheap to implement.

2 The restricted three-body problem

In this section I recall the useful features of the restricted three-body problem needed to understand the properties of the motion about L2.

Let us consider a system consisting of two bodies of respective masses M and m revolving on circular orbits around their common centre of mass. A point mass particle of negligible mass experiences the attraction of the two main bodies, but does not perturb their respective keplerian motion.

Units are so chosen as to have for unit length the distance between the two primaries, the unit of time is such that the mean orbital motion is $n = 1$, or equivalently the orbital period of the primaries is 2π. One introduces also the mass factor $\mu = m/(M + m)$ and the two masses can then be represented by $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$. In the following the big mass will be the Sun and the small mass is the Earth-Moon system. With $m/M = 1/328900.56$ one has $\mu = 1/328901.56 = 0.304042340 \times 10^{-5}$ which will be the small parameter of the problem. The centre of mass is the origin of coordinates and on the $x-$axis the two masses

are located at the abscissas,

$$x_1 = -\mu \quad (1)$$

$$x_2 = 1 - \mu \quad (2)$$

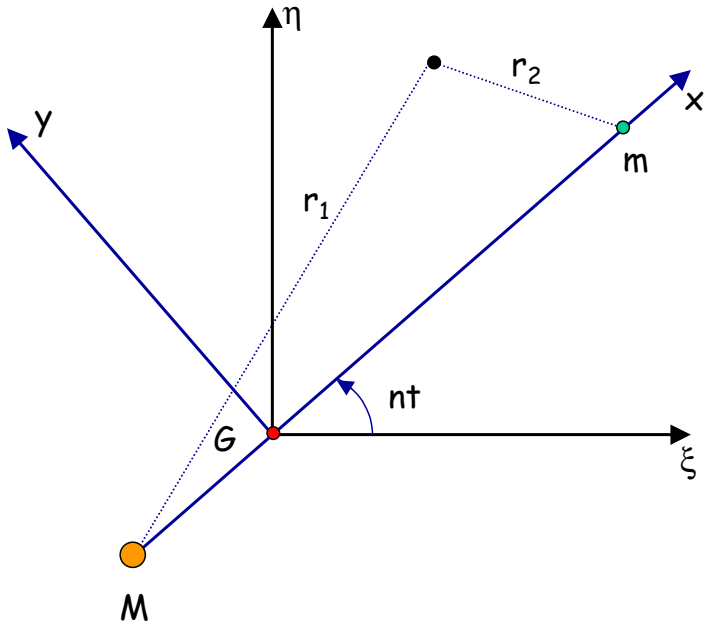


Figure 1: *Planar view of the inertial frame ($G\xi$, $G\eta$ and of the rotating frame Gx, Gy. The two primaries are aligned on the Gx axis.*

In the rotating frame (Fig. 1) the equations of motion of the massless body are given by,

$$\ddot{x} - 2\dot{y} = x + \frac{\partial U}{\partial x} \quad (3)$$

$$\ddot{y} + 2\dot{x} = y + \frac{\partial U}{\partial y} \quad (4)$$

$$\ddot{z} = \frac{\partial U}{\partial z} \quad (5)$$

where the gravitational potential at P is given by,

$$U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \quad (6)$$

with r_1 and r_2 for the distances of the two primaries from the particle as illustrated in Fig. 1.

Although this is not useful in the following it is worth mentioning that Eqs. 3–5 possess a constant of motion in the rotating frame, known as the Jacobi integral and given by,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - (x^2 + y^2) - 2U = C \tag{7}$$

The Jacobi integral is the only integral of the restricted three-body problem which means that the problem cannot be solved in closed form for arbitrary initial conditions.

3 Equilibrium positions

3.1 Circular problem

Equilibrium points (meaning relative equilibrium in the rotating frame) are found by searching positions x, y, z such that a particle initially at rest at these points undergoes no acceleration in the rotating frame. These positions are found as solutions of the following system of non linear equations,

$$x + \frac{\partial U}{\partial x} = 0 \tag{8}$$

$$y + \frac{\partial U}{\partial y} = 0 \tag{9}$$

$$\frac{\partial U}{\partial z} = 0 \tag{10}$$

The third equation with (6) implies that the equilibrium points lie in the $x - y$ plane. A famous solution of Eqs. 8–10 discovered by Lagrange as a general solution of the three-body problem consists of the three bodies revolving on circular orbits each located at a vertex of an equilateral triangle,the third body being at L4 or L5. In this present note I am only interested in the location of the collinear points, that is to say solutions of Eqs. 8–9 with $y = 0$, which is a solution of Eq. 9.

Although the problem is treated extensively in numerous textbook, it is interesting to give the details for the L2 point, which is located on the x-axis beyond the secondary in the positive direction.

Let ρ be the distance (> 0) of $L2$ from the secondary. Therefore one has,

$$x = 1 - \mu + \rho \tag{11}$$

$$y = 0 \tag{12}$$

$$z = 0 \tag{13}$$

and

$$r_1 = 1 + \rho \tag{14}$$

$$r_2 = \rho \tag{15}$$

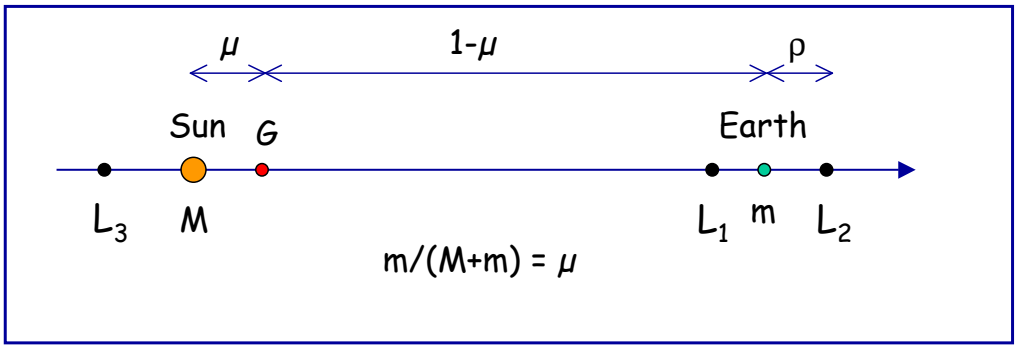


Figure 2: *Relative positions of the collinear Lagrange equilibrium points in the rotating frame. Several kind of ordering appear in the literature.*

Then by differentiating $U(x, y, z)$ with respect to x and inserting in Eq. 8 one has the equilibrium equation,

$$\frac{1-\mu}{(1+\rho)^2} + \frac{\mu}{\rho^2} = 1 - \mu + \rho \tag{16}$$

which expresses that the two attractions balance exactly the centrifugal force at the equilibrium point. After reduction, this leads to the famous quintic equation in ρ,[2]

$$\rho^5 + (3-\mu)\rho^4 + (3-2\mu)\rho^3 - \mu\rho^2 - 2\rho\mu - \mu = 0 \tag{17}$$

For the sake of completion the corresponding equations for L1 would be,

$$\rho^5 - (3-\mu)\rho^4 + (3-2\mu)\rho^3 - \mu\rho^2 + 2\rho\mu - \mu = 0 \tag{18}$$

where ρ is the distance to the secondary in the direction of the primary, while the position of L3 is given by the solution of,

$$\rho^5 - (7+\mu)\rho^4 + (19+6\mu)\rho^3 - (24+13\mu)\rho^2 + (12+14\mu)\rho - 7\mu = 0 \tag{19}$$

where $1-\rho$ is the distance of L3 from the primary.

It can be shown that for $\mu < 1$ Eq. 17 has a single real root, which is positive. For $\mu = 0$ this equation reduces to,

$$\rho^3(\rho^2 + 3\rho + 3) \tag{20}$$

in which the roots of the quadratic term are complex. Therefore for $\mu \ll 1$ the real root can be expanded in power of $\mu^{\frac{1}{3}}$. This can be done either by identification or by reversion of the expansion of μ in power of ρ from Eq. 16 with,

$$\mu = \frac{3\rho^3\left(1 + \rho + \rho^2/3\right)}{1 + 2\rho + \rho^2 + 2\rho^3 + \rho^4} \tag{21}$$

[2]The similar equation in [1] has inconsistencies in signs, following from a sign error in Eq. 2.5

This yields with[3] $\alpha = (\mu/3)^{1/3}$,

$$\alpha = \rho - \frac{1}{3}\rho^2 + \frac{1}{3}\rho^3 - \frac{80}{81}\rho^4 + \frac{371}{243}\rho^5 + O\left(\rho^6\right) \tag{22}$$

and finally for the position of L2,

$$\left\{ \begin{array}{lcl} r_2 & = & \rho \quad = \quad \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 + \frac{50}{81}\alpha^4 + \frac{43}{243}\alpha^5 + O\left(\alpha^6\right) \\ r_1 & = & 1 + \rho \end{array} \right. \tag{23}$$

In the case of the Sun+ Earth-Moon system, one finds $\alpha = 0.0100447147$ and the for the position of L2, $\rho = 0.01007824044$ or 1 507 700 km. A direct numerical solution of Eq. 17 gives the same result.

3.2 The Elliptic Problem

The actual motion of the Earth-Moon system around the Sun is slightly elliptical with an eccentricity equal to 0.0167. Hence the distance between the two primaries varies between 0.983 and 1.017 AU with a period of one sidereal year. In addition the angular motion of the rotating frame is no longer uniform. These features make the elliptical restricted three-body problem not as simple as the circular case. In fact the complete problem does not possess an integral of motion.

However the particular solutions of Lagrange are more general than the restricted problem and there are general solutions in which the three bodies move on similar ellipses about the centre of mass of the system. In particular the three bodies can stay aligned in a rotating frame revolving at the same rate as the true anomaly. It is shown in [5] that scaling the distances with the actual radius vector of the orbit of the Earth around the Sun, and making use of the true anomaly instead of the time, yield equations of motions having exactly the same form as in the case of the circular problem. The positions of the equilibrium points in the rotating-pulsating coordinates are the same as in the case of the circular problem.

This means that there is an equilibrium point in a rotating-pulsating coordinates, in which the Sun, the Earth and L2 remain aligned. But the true distance of L2 from the Earth, changes in the same proportion as the Sun-Earth distance. So as far as the location of L2 is concerned there is no major difficulty to take into account the elliptical shape of the orbit of the Earth. Applying a scale factor $1 + e\,\cos M$, where M is the mean anomaly of the Earth and e the eccentricity of the orbit, will be sufficient for the purpose of providing a reference orbit for the simulations of GAIA. Moreover the x-axis will rotate at the same rate as the Earth around the Sun, so that the three bodies keep moving aligned on this axis.

[3] Conventions and notations follow Moulton [2]. Both Brouwer and Clemence [3] and Murray and Dermott [4] use as a small parameter the mass ratio $\nu = \mu/(1-\mu)$ instead of μ, so their expansions in power of α are slightly different. One has $\beta = (\nu/3)^{1/3} = \alpha + \alpha^4 + 2\alpha^7 + O\left(\alpha^{10}\right)$

4 Eclipses at L2

A spacecraft exactly stationed at L2 will be in the shadow cast by the Earth and deprived of solar power. In principle the eclipse, while central, could be either total –if the disk of the Earth hides totally the Sun– or annular whenever the apparent solar disk is larger than that of the the Earth. It is interesting to note the nearly perfect scaled model between

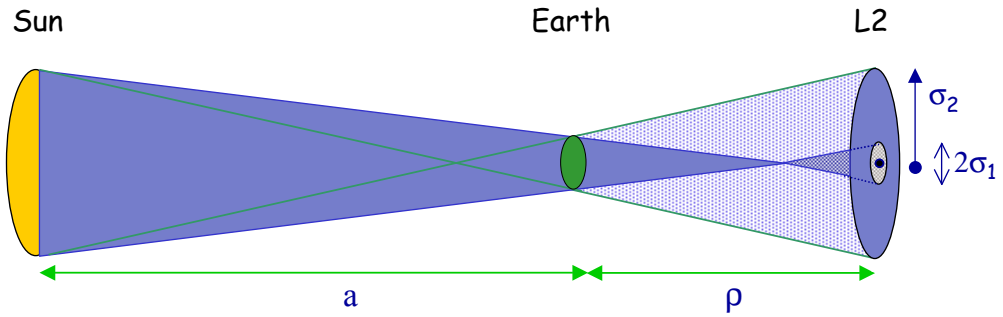


Figure 3: *Typical conditions met during a central eclipse at L2. The central eclipse is always annular with about 85 percent of the Sun obscured by the Earth.*

the system composed of the Sun, the Moon (at new Moon) and the Earth, and that with the Sun, the Earth and the point L2. Roughly the angular diameter of the Moon as seen from the Earth is the same as that of the Earth as seen from L2, or in other way, the distance of L2 from the Earth is about 4 times that of the Moon from the Earth, which also approximately the ratio of their diameter. So as the Earth is very close to the tip of the eclipse cone in the Sun-Moon-Earth system, so is L2 for the eclipse of the Sun brought about by the Earth. The diameter of the shadow is then very small. In fact the situation is such that only annular eclipses occur at L2.

Consider the average conditions to determine the eclipse configuration, with $\rho_{L2} = 1.51 \times 10^6$ km, and 1 au for the distance between the centre of the Sun and the barycentre of the Earth-Moon system. The representative geometry is sketched in Fig. 3.

The distance d_v from the Earth to the vertex of the shadow cone is given by,

$$d_v = a \frac{R_\oplus}{R_\odot - R_\oplus} \tag{24}$$

with $R_\odot$ and $R_\oplus$ respectively for the radius of the Sun and the Earth. With $a = 1$ au, one obtains $d_v = 1.385 \times 10^6$ km, more that one hundred thousand kilometres in front of L2. The radius of the shadow (positive for total eclipses, negative for annular eclipses) and of the penumbra circles at L2 are then derived from elementary geometry,

$$\sigma_1 = R_\oplus - \frac{\rho}{a}(R_\odot - R_\oplus) \tag{25}$$

$$\sigma_2 = R_\oplus + \frac{\rho}{a}(R_\odot + R_\oplus) \tag{26}$$

giving, $|\sigma_1| = 520$ km, $\sigma_2 = 13\,400$ km. So, because of the smallness of σ_1, the occurrence of a central eclipse is a very exceptional event. But even in this case, the eclipse is not total and a fraction of sunlight is still available. The average angular radius of the Sun for an observer at L2 is 15.88 arcmin and that of the Earth 14.67 arcmin. Therefore the fraction of the sunlight blocked off by the Earth is 85 percent, when atmospheric refraction is neglected.

If one wants to avoid any change in the thermal environment and take the maximum solar power on must stay outside the circle of penumbra at any time. During the motion of GAIA about L2, the distance to L2 on the x-axis could be as large as 10^5 km and one must then consider a safety margin and take the maximum penumbra circle at 1.6×10^6 km, that is to say a radius to 14 000 km.

Regarding the amount of sunlight, the transition from the annular eclipse to full sunlight is gradual, and it would not be dramatic to go through the penumbra for a few hours, as long as the sun can provide enough power to the solar cells. The situation is shown in Fig. 4 which gives the fraction of sun power available as a function of the distance of the spacecraft from L2 in a plane perpendicular to the Sun-Earth line. Outside a radius of 6 000 km there are more than 50 percent of the solar power available.

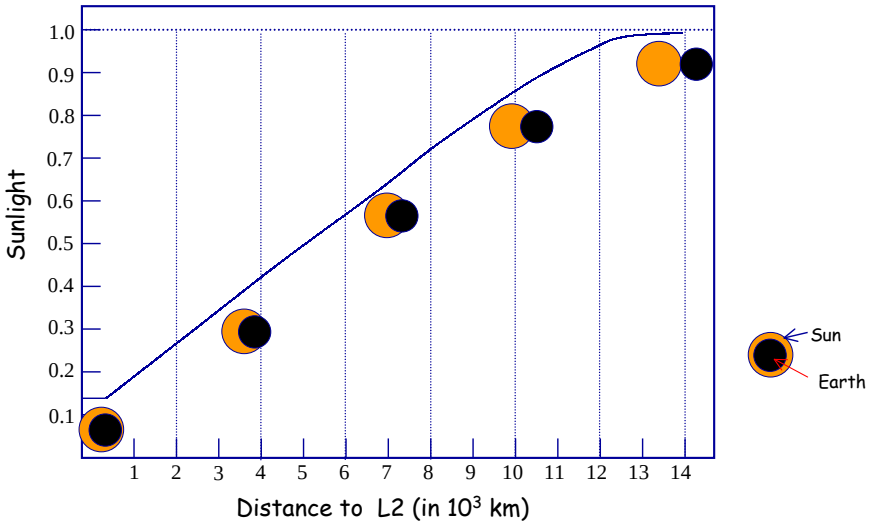


Figure 4: *Fraction of the solar power available during the annular eclipse at L2 (distance to L2 < 500 km) and during the partial eclipse, until the distance reaches 14 000 km at the end of the eclipse. The Sun and Earth disks are sketched in the plot at different phases.*

5 Small motion about L2

5.1 Equations

It is not enough to know that there is an equilibrium point at L2, since this tells us nothing about its stability. It is needed that a spacecraft placed at L2 remains in its vicinity despite small displacements and non zero speed. By restricting to the linear stability one can apply the general theory of infinitesimal motion about the equilibrium point to analyse the linear stability.

Let $x_0 = \rho, y_0 = 0, z_0 = 0$ the coordinates of L2 and the small displacement about L2 be represented by (X, Y, Z) such that $x = x_0 + X$, $y = y_0 + Y$, $z = z_0 + Z$. To the first order in X, Y, Z, one can expand the equations of motion about L2 and using Eqs. 8– 10 the equations for the small motions read,

$$\ddot{X} - 2\dot{Y} = X + \left(\frac{\partial^2 U}{\partial x^2}\right)_0 X \tag{27}$$

$$\ddot{Y} + 2\dot{X} = Y + \left(\frac{\partial^2 U}{\partial y^2}\right)_0 Y \tag{28}$$

$$\ddot{Z} = \left(\frac{\partial^2 U}{\partial z^2}\right)_0 Z \tag{29}$$

where the derivatives are evaluated at L2.

With

$$U(x, y, z) = \frac{1-\mu}{((x+\mu)^2 + y^2 + z^2)^{1/2}} + \frac{\mu}{((x-1+\mu)^2 + y^2 + z^2)^{1/2}} \tag{30}$$

one gets,

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_0 = 2\Omega \tag{31}$$

$$\left(\frac{\partial^2 U}{\partial y^2}\right)_0 = -\Omega \tag{32}$$

$$\left(\frac{\partial^2 U}{\partial z^2}\right)_0 = -\Omega \tag{33}$$

with,

$$\Omega = \frac{1-\mu}{(1+\rho)^3} + \frac{\mu}{\rho^3} \tag{34}$$

This leads to the equations for the small motions,

$$\ddot{X} - 2\dot{Y} = (1 + 2\Omega)\, X \tag{35}$$

$$\ddot{Y} + 2\dot{X} = (1 - \Omega)\, Y \tag{36}$$

$$\ddot{Z} = -\Omega\, Z \tag{37}$$

For $\mu \ll 1$ on has,

$$\Omega = 4 - 6\alpha + 8\,\alpha^2 - \frac{50}{3}\,\alpha^3 + \frac{518}{27}\,\alpha^4 + O\left(\alpha^5\right) \tag{38}$$

and numerically $\Omega = 3.940522185$.

5.2 Solution for Z

The last equation of motion is independent of the first two and can be treated separately. As we have $\Omega > 0$ one can put $\omega = \Omega^{1/2}$ and the solution is,

$$Z = \alpha_z\,\cos(\omega t) + \beta_z\,\sin(\omega t) \tag{39}$$

where α_z and β_z are constants to be determined from the initial conditions. Therefore the motion perpendicular to the ecliptic is periodic and sinusoidal and,

$$\omega = 2\left(1 - \frac{3}{4}\alpha + \frac{23}{32}\alpha^2 - \frac{593}{384}\alpha^3 + \frac{54281}{55296}\alpha^4\right) + O\left(\alpha^5\right) \tag{40}$$

For the Earth-Sun L2, $\omega = 1.985074856$, meaning that the period of the z-motion is close to six months, more precisely $P_z = 184.00$ days.

5.3 Solution for X, Y

Eqs. 35-36 are coupled linear equations. Again one applies the general theory by searching particular solutions of the form $\alpha_x\,e^{-\lambda t}$ and $\alpha_y\,e^{-\lambda t}$. This is inserted in (35)-(36) and leads to,

$$\begin{aligned} \left[\lambda^2 - (1+2\Omega)\right]\alpha_x - 2\lambda\alpha_y &= 0 \qquad (41)\\ 2\lambda\alpha_x + \left[\lambda^2 - (1-\Omega)\right]\alpha_y &= 0 \qquad (42) \end{aligned}$$

which has a solution provided λ is an eigenvalue of the linear application or, equivalently, provided λ is one of the roots of the characteristic equation,

$$\begin{vmatrix} \lambda^2 - (1+2\Omega) & -2\lambda \\ +2\lambda & \lambda^2 - (1-\Omega) \end{vmatrix} = 0 \tag{43}$$

giving,

$$\lambda^4 + \lambda^2\,(2-\Omega) - (2\Omega^2 - \Omega - 1) = 0 \tag{44}$$

The nature of the motion depends on the character of the roots of Eq. 44. For $\mu \ll 1$, we have with (38) $\Omega \approx 4$, so the term independent of λ in (44) is negative and the equation in λ^2 has two real roots of opposite sign, λ_+^2, λ_-^2. Therefore the equation in λ has two real roots λ and $-\lambda$ and two purely imaginary roots $i\sigma$ and $-i\sigma$. The real roots generate

an exponentially growing or shrinking motion, whereas the imaginary roots produce a bounded periodic motion.

From (44) one calculates the two roots,

$$\lambda_+^2 = \frac{\Omega - 2 + (9\Omega^2 - 8\Omega)^{1/2}}{2} \tag{45}$$

$$\lambda_-^2 = \frac{\Omega - 2 - (9\Omega^2 - 8\Omega)^{1/2}}{2} \tag{46}$$

which can be expanded in power of α for $\mu \ll 1$ with (38) as,

$$\lambda_+^2 = (1 + 2\sqrt{7}) - \left(3 + \frac{24\sqrt{7}}{7}\right)\alpha + \left(4 + \frac{887}{196}\sqrt{7}\right)\alpha^2 - \left(\frac{25}{3} + \frac{9755}{1029}\sqrt{7}\right)\alpha^3 + O\left(\alpha^4\right)$$

$$\lambda_-^2 = (1 - 2\sqrt{7}) - \left(3 - \frac{24\sqrt{7}}{7}\right)\alpha + \left(4 - \frac{887}{196}\sqrt{7}\right)\alpha^2 - \left(\frac{25}{3} - \frac{9755}{1029}\sqrt{7}\right)\alpha^3 + O\left(\alpha^4\right)$$

Taking now the square root one obtains the eigenfrequencies as,

$$\lambda_1 = \left(1 + 2\sqrt{7}\right)^{1/2}\left[1 - \frac{3 + 24/\sqrt{7}}{1 + 2\sqrt{7}}\,\frac{\alpha}{2} + \frac{1247 + 1447/\sqrt{7}}{(1 + 2\sqrt{7})^2}\,\frac{\alpha^2}{56}\right] + O\left(\alpha^3\right) \tag{47}$$

$$\lambda_2 = -\lambda_1 \tag{48}$$

$$\sigma_1 = \left(2\sqrt{7} - 1\right)^{1/2}\left[1 + \frac{3 - 24/\sqrt{7}}{2\sqrt{7} - 1}\,\frac{\alpha}{2} + \frac{1247 - 1447/\sqrt{7}}{(2\sqrt{7} - 1)^2}\,\frac{\alpha^2}{56}\right] + O\left(\alpha^3\right) \tag{49}$$

$$\sigma_2 = -\sigma_1 \tag{50}$$

Numerically one finds, with a complete numerical solution of (44)

$$\lambda = 2.484316720 \tag{51}$$

$$\sigma = 2.057014191 \tag{52}$$

5.4 The periodic solution

With the knowledge of the nature of the eigenvalues it is interesting to rewrite the form of the general solution as follows,

$$X = a_1\, e^{\lambda\, t} + a_2\, e^{-\lambda\, t} + a_3\, \cos(\sigma\, t) + a_4\, \sin(\sigma\, t) \tag{53}$$

$$Y = b_1\, e^{\lambda\, t} + b_2\, e^{-\lambda\, t} + b_3\, \sin(\sigma\, t) + b_4\, \cos(\sigma\, t) \tag{54}$$

where the a_i, b_i satisfy identically (35) or (36). By inserting (53)-(54) in either of (35) - (36) one links each b_i to a_i as,

$$b_1 = \frac{\lambda^2 - (1+2\Omega)}{2\lambda}\, a_1 \tag{55}$$

$$b_2 = -\frac{\lambda^2 - (1+2\Omega)}{2\lambda}\, a_2 \tag{56}$$

$$b_3 = -\frac{\sigma^2 + (1+2\Omega)}{2\sigma}\, a_3 \tag{57}$$

$$b_4 = \frac{\sigma^2 + (1+2\Omega)}{2\sigma}\, a_4 \tag{58}$$

$$\tag{59}$$

Then the initial conditions in X_0, Y_0, $\dot{X}_0$, $\dot{Y}_0$ would determine the four a_i with,

$$X_0 = a_1 + a_2 + a_3 \tag{60}$$

$$Y_0 = b_1 + b_2 + b_4 \tag{61}$$

$$\dot{X}_0 = \lambda(a_1 - a_2) + \sigma\, a_4 \tag{62}$$

$$\dot{Y}_0 = \lambda(b_1 - b_2) + \sigma\, b_3 \tag{63}$$

However we are concerned now with a solution in which the exponential terms have a zero amplitude, that is to say one wishes to impose $a_1 = a_2 = 0$, and with (55) $b_1 = b_2 = 0$. In this case only two initial conditions remain arbitrary and one has for example ,

$$a_3 = X_0 \tag{64}$$

$$a_4 = Y_0/\beta \tag{65}$$

where

$$\beta = \frac{\sigma^2 + 1 + 2\Omega}{2\sigma} = 3.18722929 \tag{66}$$

or in power of the small parameter α,

$$\beta = \left(2\sqrt{7}+5\right)^{1/2} - \frac{27}{14}\,\frac{7+\sqrt{7}}{\left(2\sqrt{7}-1\right)^{3/2}}\alpha + \frac{9}{392}\,\frac{604\sqrt{7}+1939}{\left(2\sqrt{7}-1\right)^{5/2}}\alpha^2 + O\left(\alpha^3\right) \tag{67}$$

The initial velocity is linked to the initial position to get rid of the exponential terms with,

$$\dot{X}_0 = \sigma\, a_4 = \sigma\,\frac{Y_0}{\beta} \tag{68}$$

$$\dot{Y}_0 = \sigma\, b_3 = -\sigma\beta\, X_0. \tag{69}$$

which means that once the initial position is selected, the initial velocity cannot be chosen freely to ensure that the solution is periodic.

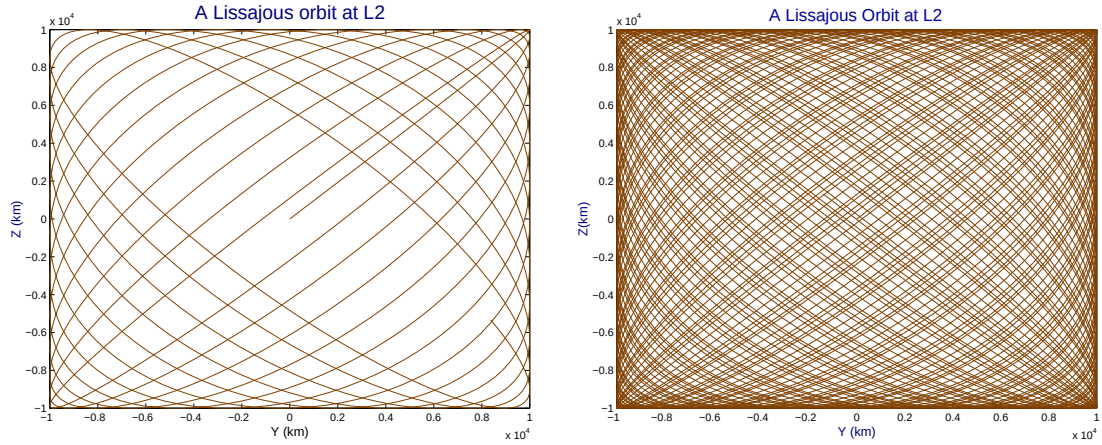


Figure 5: *Lissajous orbit about the L2 point over 6 years (left) and 35 years (right). The scale in kilometre is arbitrary.*

This finally leads to the expression of the periodic solution,

$$X = X_0 \cos\sigma t + \frac{Y_0}{\beta} \sin\sigma t \tag{70}$$

$$Y = -\beta X_0 \sin\sigma t + Y_0 \cos\sigma t \tag{71}$$

In the plane $x - y$ the trajectory is an ellipse with the major axis along the y-axis and the ratio between the two axes is equal to β. The period of the motion in years is $1/\sigma$ or 177.566 days.

6 The Lissajous orbit

6.1 General properties

The periods of the motion in the $x - y$ plane and that of the motion in the perpendicular direction, are not identical, although both are close to six months. It follows that the 3-D motion is not periodical. If we look at the projection of the motion in the $y - z$ plane, we have with a possible choice of initial conditions and origin of time,

$$Y = a \sin 2\pi\sigma t \tag{72}$$

$$Z = b \sin 2\pi\omega t \tag{73}$$

with, as found earlier, $\sigma = 2.057014191$ and $\omega = 1.985074856$ and the unit of time is now the sidereal year (so the factors 2π have been added). Since the ratio of the two frequencies is not rational (at least from their very definition), the trajectory will fill a rectangle of sides $2a$, $2b$ drawn in the $y - z$ plane, as illustrated in Fig. 5. However, if the motion is not periodic it can be stated from a general theorem of dynamical systems, that there are

infinitely many t_i, where the moving point will cross a circle of radius ϵ around the origin. In other way, the equation $\left(Y(t)^2 + Z(t)^2\right)^{1/2} < \epsilon$ has infinitely many solutions in t.

If the magnitudes of the in-plane and out-of-plane motions are large enough so that the non-linear effects (the terms in X^2, Y^2, XY neglected in Eqs. 35-36) produce eigenvalues that can be equal, that is to say $\sigma/\omega \to 1$ as a, b increase. In this case the 3-D orbit becomes periodic and is known as a halo orbit [6]. This situation is not considered in this note, although with the size of the orbit needed to escape eclipses, one comes close to the non-linear regime. However, the linear approximation is sufficient for the purpose of this work motivated by the needs of the GAIA simulations and for providing insights on the Lissajous motion.

6.2 The quasi-periodic motion

Let $T_y = 1/\sigma$ (= 177.57 days) and $T_z = 1/\omega$ (= 184.00 days) be the periods[4] of $Y(t)$ and $Z(t)$. Were the ratio σ/ω a rational number p/q, the motion would be periodic with a period $P = pT_y = qT_z$ and it will take about half this time before the first return to the origin. If σ/ω is close to a rational number, there is no strict return to the origin, but a passage in the vicinity of the origin, all the more closer as the distance between the frequency ratio and a rational number is small.

A general theorem of Dirichlet tells us that any irrational number ξ can be approximatesd by a rational number p/q such that

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{q^2} \tag{74}$$

and that there are infinitely many solutions to the above inequality. In addition, the theory of the best approximations of irrational numbers by rational fractions provides us with an infinite sequence p_n, q_n fulfilling this criterion with [7],

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2} \tag{75}$$

where the p_n/q_n are the convergents in the expansion of α in continued fractions. This tells us that good rational approximations of σ/ω do exist and consequently these approximations will lead to a near return to the origin after a certain lapse of time. The convergents have the nice property that there is no fraction p/q with $q < q_n$ such that

$$|p - q\,\alpha| < |p_n - q_n\,\alpha| \tag{76}$$

meaning the convergents have the 'less complexity' achievable to approximate an irrational number (or in practice a rational with a very large denominator).

The best approximations for the small motions about L2 are listed in Table 1. The periods in years correspond to p_n/σ or q_n/ω. For example one reads that 29 periods in

[4] One must be careful that the motions are periodic in the rotating frame ; the periods are then synodic periods.

$Y(t)$ corresponds almost exactly to 28 periods in the Z-motion and that the resulting quasi period is 14.1 years.

Table 1: *Best rational approximations of* $\sigma/\omega = 1.0362401093$ *for the small motion of L2 for the Sun and the Earth–Moon system.*

p_n	q_n	p_n/q_n	$\lvert p_n/q_n - \alpha\rvert$	$\lvert p_n - q_n\alpha\rvert$	Period (yrs)
2	2	1.0000000000	3.6E-02	7.2E-02	1
27	26	1.0384615384	2.2E-03	5.8E-02	13.1
28	27	1.0370370370	8.0E-04	2.2E-02	13.6
29	28	1.0357142857	5.3E-04	1.4E-02	14.1
57	55	1.0363636363	1.2E-04	6.8E-03	27.7
143	138	1.0362318840	8.2E-06	1.1E-03	69.5
772	745	1.0362416107	1.5E-06	1.1E-03	375.3
915	883	1.0362400906	2.1E-08	1.9E-05	444.8

6.3 Returns to origin

What matters now, is how far from the origin these returns would be over a duration less than $\sim$ 10 years, the maximum duration of the GAIA flight. More precisely one wishes to avoid the crossing of the penumbra area, which has a radius of 14 000 kilometres. So starting from the origin (this is not essential for the argument), the question is for how long can we stay outside this area and what should be the amplitude of the Y and Z motion that would meet this requirement ?

The distance to the origin is given by $\left(Y(t)^2 + Z(t)^2\right)^{1/2}$ or,

$$\rho^2 = a^2 \sin^2 2\pi\sigma\, t + b^2 \sin^2 2\pi\omega\, t \tag{77}$$

and is close to zero when one has simultaneously

$$\sigma\, t \simeq p/2 \tag{78}$$
$$\omega\, t \simeq q/2 \tag{79}$$

where p, q are arbitrary integers. Therefore a good rational approximation p/q of σ/ω will lead to a near return to origin.

Let $\sigma/\omega = p/q + \epsilon$ be a particular approximation and, just for symmetry reasons denote also $\omega/\sigma = q/p - \eta$, giving $\eta \sim (q/p)^2\, \epsilon$.

One then introduces the times close to the return to the origin as,

$$t_1 = \frac{p}{2\sigma} \tag{80}$$

$$t_2 = \frac{q}{2\omega} \tag{81}$$

One has $Y(t_1) = 0$, $Z(t_1) \neq 0$ and $Y(t_2) \neq 0$, $Z(t_1) = 0$ and $t_1 \sim t_2$. At t_1 the phase of the Z- motion is close to 0 or to π with,

$$2\pi\,\omega t_1 \simeq \pi\,(q - p\eta) \tag{82}$$

leading to $z = Z(t_1) \simeq (-)^{q+1} b\,\pi\,p\,\eta$. With the same argument applied to t_2 and $Y(t)$ this gives $y = Y(t_2) \simeq (-)^p\,a\,\pi\,q\,\epsilon$. As illustrated in Fig. 6, the minimum distance to the origin in the vicinity of t_1 and t_2 is given by,

$$\rho = \frac{|y\,z|}{\sqrt{y^2 + z^2}} \tag{83}$$

yielding,

$$\rho = \frac{\pi\,ab\,\eta\,\epsilon pq}{(a^2q^2\epsilon^2 + b^2p^2\eta^2)^{1/2}} \tag{84}$$

giving a much simpler expression whenever $a = b$ and $\sigma/\omega \simeq 1$,

$$\rho = \frac{\pi\,a}{\sqrt{2}}\,|q\,\epsilon| \tag{85}$$

The minimum distance is scaled by the amplitude a and by the residual $q\epsilon = p - q\alpha$.

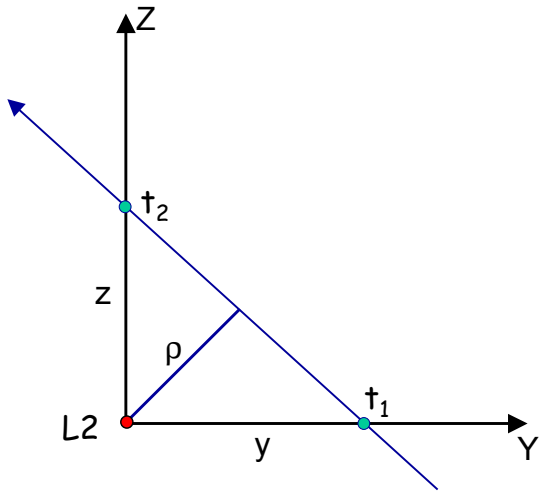


Figure 6: *Close-up view of the trajectory about L2 after a near integral or half integral numbers of periods in Y and Z.*

Table 2: *Theoretical minimum distance to the origin for a Lissajous motion about L2 for an amplitude in Y and Z equal to* 10^4 *km.*

p_n	q_n	$q_n\,\epsilon$	Distance km	Duration yrs
27	28	5.8E-02	1300	6.5
28	27	2.2E-02	490	6.8
29	28	1.4E-02	310	7.05
57	55	6.8E-03	150	13.85
143	138	1.1E-03	25	34.75

A plot of $\rho(t)$ is shown in Fig. 7 over 40 years and by using $a = b = 10^4$ km. The distances to the origin for the three convergents are in perfect agreement with the computed values of Table 2 based on Eq. 85. The smallest distances get smaller as the rank of the convergents increases, since the criteria for the best approximation is precisely the decrease of $q_n\,\epsilon$. The other minimum correspond to good rational approximations, but not the best approximations.

6.4 Size of the orbit for GAIA

Now if one wants to avoid the eclipse region during a certain interval of time, it remains only to determine the amplitude of the two motions, so that the smallest distance is always larger that the circle of umbra or penumbra. The only relevant case for GAIA is the 7-year duration, corresponding to the fraction 29/28 in Table 2. If this eclipse season is successfully skipped, there is another period of 7 years without eclipses, which is much above the requirement for GAIA. Therefore one can start by considering that one wishes to avoid the first eclipse season. The size of the penumbra has been found close to $s = 14\,000$ km, which gives with Eq. 85,

$$a > \frac{\sqrt{2}}{\pi}\,\frac{s}{q_n \epsilon} \tag{86}$$

or numerically, $a > 450\,000$ km. As a consequence the motion along the x-axis has an amplitude of 140 000 km from Eqs. 66-70-71. The corresponding Lissajous orbit is plotted in Fig. 8, with a close-up view showing the closest passage after 7.05 years. No closer passage would occur before the next eclipse season around $t \sim 14$ years.

However there is some more freedom to select a and b as nothing forces us to impose $a = b$. The amplitude of the motion perpendicular to the ecliptic can be different from the amplitude in the ecliptic. Take for the scale of the orbit the size of the diagonal of

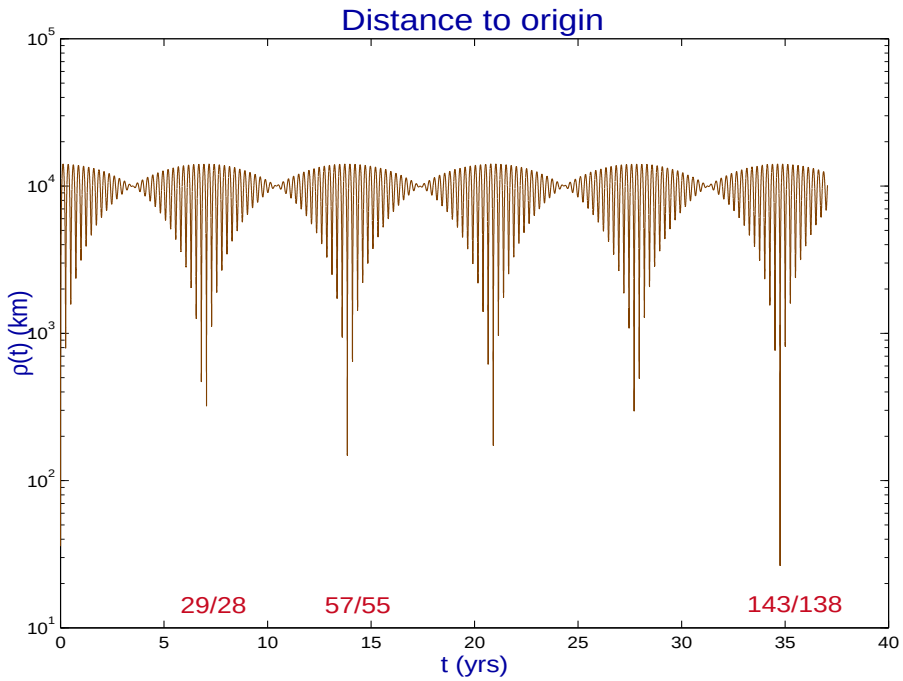


Figure 7: *Distance to the origin for the Lissajous motion for the amplitude* $a = b = 10^4 km$. *The best rational approximations of the periodic in- and out-of-plane motions are indicated above the x-axis.*

the rectangle in which the Lissajous orbit in inscribed and call $d^2 = a^2 + b^2$. At a given approximation, that is to say for a particular choice of p_n, q_n one would like to have the largest minimal distance ρ given by Eq. 84. So the problem is to search the least value of this equation with the constraint that $a^2 + b^2 = d^2$ is given. Using Lagrange multipliers one finds that the best choice corresponds to $a^2/b^2 = q_n/p_n$. Therefore in the present case, with the two frequencies quite alike, this does not make much difference with the adopted simpler solution $a = b$.

The value found to avoid eclipses is strongly dependent on the closeness of the rational approximation to the ratio of the proper frequencies. The closer to a rational, the smaller the distance to the origin after ~ 7 years. But the approximation is itself very sensitive to the precise numerical values of the frequencies. With the ratio $\sigma/\omega = 1.0362401$ one gets $a = 450\,000$ km whereas $350\,000$ are sufficient with a ratio equal to 1.036340. Taking for example the Earth mass instead of the Earth-Moon mass may change significantly the frequencies and the closeness of the rational approximation. So something around 400 000 km is needed to skip the first eclipse season and to have a motion without eclipse for about 14 years. Only a full numerical integration of the problem could provide the correct amplitude and periods. Investigations of the periodic orbit of the Hill problem (when $\mu \to 0$) by M.Hénon [8] shows that the frequency of the X and Y motion increases, although very slowly, with the amplitude of the motion (about 0.2 percent for an amplitude in Y of $400\,000$ km). The distortion of the orbit is more significant, changing gradually

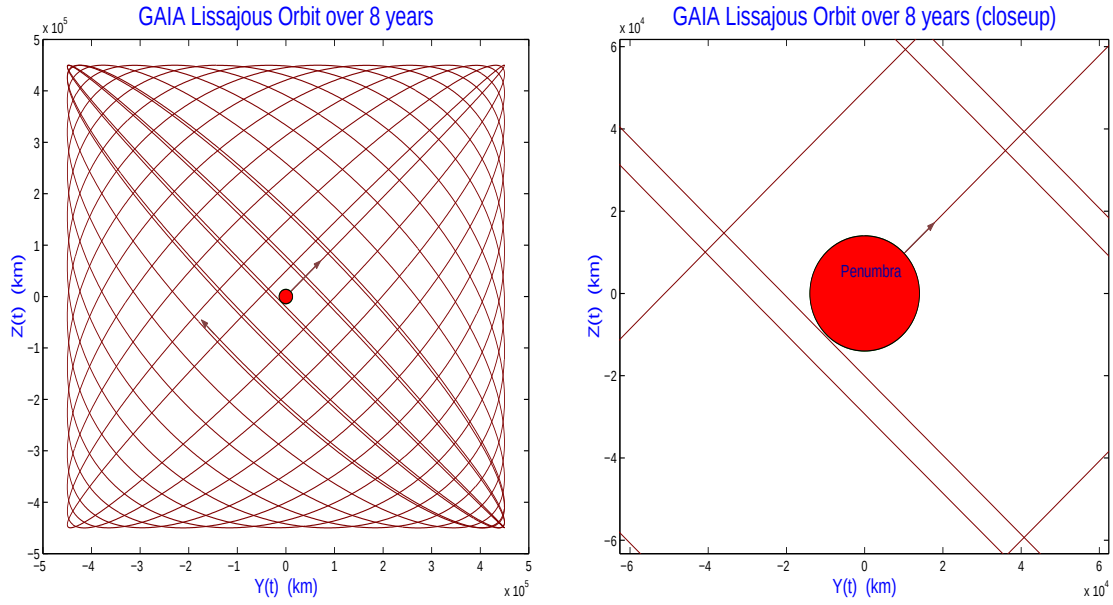


Figure 8: *The Lissajous orbit for* GAIA *over 8 years. The projected penumbra circle is drawn at the centre of the plot. Non linear effects in the motion are disregarded. On the right the central part of the Lissajous orbit over 8 years. No passage in the penumbra circle occurs during this period and during the next 7 years.*

from an ellipse in the $X - Y$ plane to a kidney like closed curve. A good illustration of a more realistic Lissajous orbit derived from numerical integration can be seen in the Study Report on p.222. [9].

However so large an amplitude is not needed for GAIA. One sees in Fig. 7 that there is no approach to the origin closer than 1000 km over a period of ~ 6.5 years, while the next two minimums at 6.8 and 7 years are much deeper. This minimum is linked to the convergent 27/26 with $q\epsilon = 5.8 \times 10^{-2}$. Therefore with Eq. 86 one finds that starting with an amplitude of $\sim 110\,000$ km is sufficient to maintain the spacecraft outside the penumbra for a little more than 6 years(one must shift the origin of time by 3 months to avoid the first eccentric loop of the Lissajous curve which enters the penumbra). The associated Lissajous orbit is plotted in Fig. 9, with a close-up view showing the closest passage after 6.3 years. Two closer passage will take place within the next six months.

This seems a reasonable compromise which has the advantage of maintaining the Sun-satellite-Earth angle to a value less than 4 degrees, in the range of the maximum efficiency of the communication antenna. But larger angles, up to 15 degrees are acceptable if a very large orbit proved to be a better solution. Then the orientation of the spin axis would have also to be adjusted if the scan obliquity had to be kept very precisely at a fixed angle (50 or 55 degrees for example). For the simulation work I suggest maintaining a total decoupling between the orbit about L2 and the scanning law, even though a different approach might be followed in the real mission.

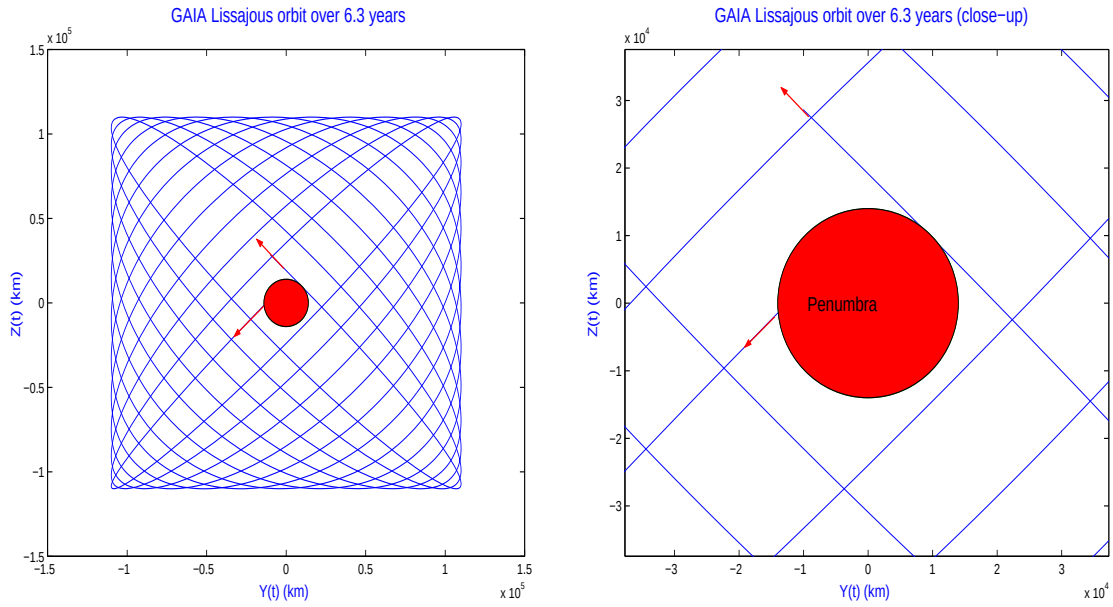


Figure 9: *A Lissajous orbit for* GAIA *over 6.3 years. The projected penumbra circle is drawn at the centre of the plot. The central part is seen in the right panel. A partial eclipse will occur at the next passage 3 months later.*

7 Reference orbit for GAIA

For the purpose of the simulation of the mission one needs an orbit which includes most of the features of the true motion about L2 such that :

- A correct order for the size of the loops around L2. This is required for the modelling of the solar system observations and to have a realistic angle between the direction of the Earth and that of the Sun.
- The location of L2 with respect to the barycentre of the solar system must be properly accounted for.
- The simulated motion must include the two main frequencies found in the framework of the restricted problem.
- The velocity must be also properly represented by the simulated orbit to test the aberration correction and, possibly, its effect in the spectro data.

7.1 Motion of L2

The motion of L2 with respect to the Sun is, within a scale factor, the same as the motion of the Earth. (see Sect.3.2). For the simulation a rather simple model of the Earth

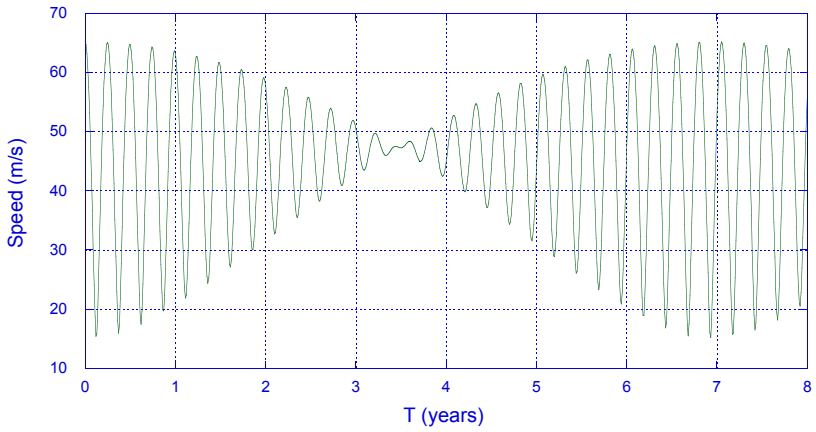


Figure 10: *Orbital speed in m/s relative to L2 for the* GAIA *Lissajous orbit (Eqs. 87) over 8 years.*

motion can be adopted, limited to the keplerian terms expanded to the second order in eccentricity. A more complex model is easily available without additional complexity in term of computation. In general such a motion will be better described in the ecliptic and mean equinox of date. However for GAIA one should express the final result in the ecliptic and origin of J2000. Finally one also needs to refer this motion to the barycenter of the solar system. Practical algorithms will be provided in a subsequent note.

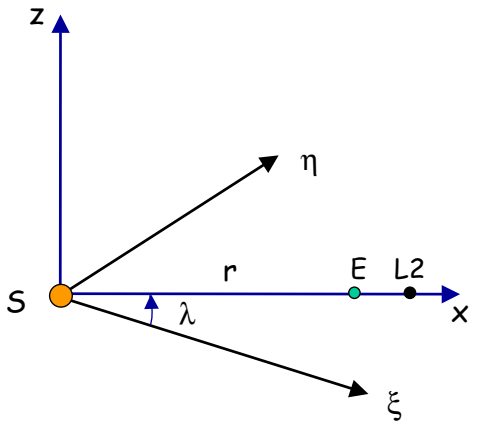


Figure 11: *Reference frame for the description of the Earth with respect to the Sun. The motion takes place in the ecliptic and* λ *is the longitude referred to the equinox of the date.*

7.2 Motion about L2

Following the conclusions of the previous sections of this note, one can take, for the small motions about L2,

$$\begin{aligned} X(t) &= -35\,000\cos(2\pi\sigma t) \\ Y(t) &= 110\,000\sin(2\pi\sigma t) \\ Z(t) &= 110\,000\sin(2\pi\omega t) \end{aligned} \tag{87}$$

where the amplitudes are in kilometers and the sideral year is the unit of time. The minus sign for $X(t)$ follows from Eqs. 70–71.

In the simulation, it is not stringent to avoid eclipses, which gives us total freedom for the origin of time. What matters is simply the geometry of the Lissajous curve from which one can properly linked observations carried out from the spacecraft to the barycenter of the solar system.

The speed of the motion relative to L2 is shown in Fig 10 and gives rise to periodic aberration of ~ 1 mas, directly proportional to the size of the orbit.

The final step consists of referring the relative motion is the non-rotating frame ($S\xi\eta Z$ in Fig. 11) and to transfer the origin at the barycenter of the solar system.

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