The Solar parallax with the transit of Venus

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Abstract

In this note I show how one can derive the timing of the four contacts of Venus with the sun disk as a function of the location of the observer and of the solar parallax with a relatively simple approximate formula whose accuracy is sufficient to process the observations of a network of observers. Two models are discussed : one, linear in the coordinates of the observer, has an accuracy of 5 to 10 s, and a quadratic one with an accuracy of 0.2 s. The expressions can then be used to determine the parallax either from the instant of the contact or from the duration of the transit between the second and third contacts. The method is applied to the historical passage of 1769 for the five stations that produced timings for the two internal contacts and is also illustrated one a worked out example for the passage of 2004.

1 Introduction

The passages of Venus on the solar disk have been used during the four previous historical events (1761, 1769, 1874 and 1882) to attempt (as suggested by E. Halley in different reports from 1677, upon his return from St Helena where he recorded the a transit of Mercury) to determine the solar parallax and then the astronomical unit. In his last paper on the subject in 1716 this great astronomer gave the mathematical details of the method and urged the astronomers to prepare themselves for the next two passages of 1761 and 1769 by planning observations at widely scattered locations to determine accurately *the interval from internal tangency of Venus's disk with the Sun's edge at ingress to internal tangency at egress*, that is the interval of time between second and third contacts.

The passage of June 2004 has provided a unique opportunity to repeat this measurements with modern techniques of timing and time transfer, allowing laymen to take part in a challenging experiment. Although the issue is no longer to determine the mean distance between the Earth and the Sun, perfectly ascertained today from radar measurements on the surface of the inner planets or from the doppler tracking of solar system orbiters, this remains a very valuable exercise to introduce students, schoolchildren and the general public to some aspect of the scientific method, let alone the direct relation of the event with some glorious pages of the history of astronomy.

However, if the principles are very simple (the length of the chord drawn by Venus changes with the position of the observer on the Earth), the details are more tricky and not liable to a simple explanation. Among the numerous sites available on Internet, most indicate very qualitatively how the parallax is obviously related to the transit observed from different places, but none details the actual model at the level of the measurement accuracy.

In the following I propose to bridge this gap by supplying a derivation of the difference of the instants of the contacts and that of the whole duration of the phenomena when observed at two different locations on the Earth. The formulas can be reduced in numbers leaving as only variable quantities the longitude and latitude of the observers and the ac-

tual solar parallax. From a comparison between the computed values based on an assumed mean Sun-Earth distance and the observed values obtained by two widely separated observers, one can quickly obtain a true estimate of that distance, with an accuracy quite equivalent to that achieved in the XVIIIth century. The very last section summarises just what is needed for the 2004 transit, for those looking for practical formulas without being interested in the technical derivation.

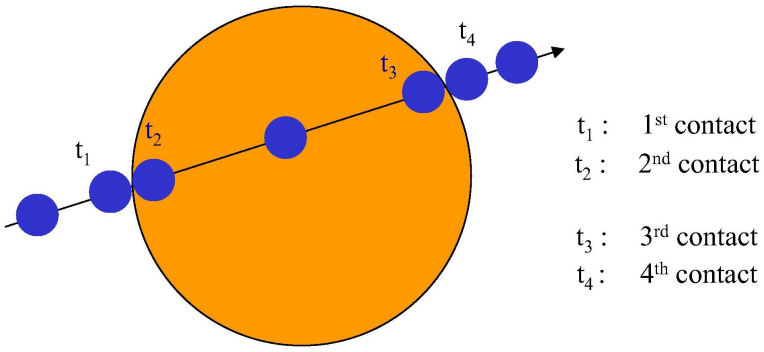


Figure 1: *The four contacts of Venus during the transit on the solar disk. Only the two inner contacts can be recorded.*

2 Geocentric contacts

In the following one considers that the geocentric timings of the four contacts have been previously computed. They have no practical importance to determine the parallax since one compares only the observations performed by two observers on the Earth. The geocentric timings are common to both observers and serve only as a reference point. They cancel out in the difference between the two actual topocentric timings.

One denotes respectively by $t_i, i = 1, 2, 3, 4$ the geocentric timings of the first external contact, the first internal contact, the second internal contact and the second external contact as illustrated in Fig. 1. One also refers the geocentric position of the centre of Venus with respect to the Sun by the rectangular coordinates on the the tangent plane with origin at the centre of the Sun by X, Y, where X is directed toward the increasing right ascension and Y to the North. If $\alpha_\odot, \delta_\odot$ and α_v, δ_v are respectively the equatorial coordinates of the Sun and Venus at any time during the transit, one has the following relationships between

the projected cartesian coordinates and the the equatorial coordinates,

$$X = \frac{\cos\delta_v \, \sin(\alpha_v - \alpha_\odot)}{\sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot)} \quad (1)$$

$$Y = \frac{\sin\delta_v \cos\delta_\odot - \cos\delta_v \sin\delta_\odot \cos(\alpha_v - \alpha_\odot)}{\sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot)} \quad (2)$$

where the denominator of these equations is related to the arc length ψ between the centre of the Sun and that of Venus, with

$$\cos\psi = \sin\delta_\odot \sin\delta_v + \cos\delta_\odot \cos\delta_v \cos(\alpha_v - \alpha_\odot) \quad (3)$$

For the following it is important to remark that the position of Venus relative to the Sun is the same as the difference of the geocentric unit vectors $\mathbf{p}$ and $\mathbf{s}$ giving respectively the directions of Venus and the Sun and that X and Y are the components of this vector projected on the tangent plane.

Eqs. 1-2 can be expanded in powers of $\Delta\alpha$ and $\Delta\delta$ to give,

$$X = \Delta\alpha \cos\delta - \sin\delta \Delta\delta\Delta\alpha + O(\Delta\alpha^3, \Delta\delta^3) \quad (4)$$

$$Y = \Delta\delta + \frac{1}{4}\sin 2\delta \Delta\alpha^2 + O(\Delta\alpha^3, \Delta\delta^3) \quad (5)$$

The second order terms are required here to predict the timing to one second of time since at the time of the contacts $\Delta\alpha, \Delta\delta \simeq \theta \sim 0\overset{\circ}{.}25$, where θ is the angular solar radius. The typical relative speed of Venus on the solar disk during a transit is $240''$ per hour, or 0.07 $''$/s. Then to predict the timing with an error less than one second of time the relative position of the two bodies must be known to better than $0\,''.05$. The second order terms in Eqs. 4 have a magnitude of $\theta^2 \sim 4''$ whereas the third order terms $\theta^3 \sim 0''.01$ can be neglected. Omitting the terms of second order may lead to an error in the timing as large as 6 seconds of time. With a numerical program, it is advisable to keep the rigourous expressions of Eqs. 1 instead of the power expansions.

However, in single precision, these expressions are not very good for numerical calculation as they are equivalent to compute a small arc on the sky from its cosine. So it is recommended in actual programming to use the alternative and numerically more accurate following expressions,

$$\sin^2\frac{\psi}{2} = \sin^2\frac{\delta_v - \delta_\odot}{2} + \cos\delta_v \cos\delta_\odot \sin^2\frac{\alpha_v - \alpha_\odot}{2} \quad (6)$$

and for the numerator of Eq. 2

$$\sin(\delta_v - \delta_\odot) + 2\cos\delta_v \sin\delta_\odot \sin^2 \frac{\alpha_v - \alpha_\odot}{2} \tag{7}$$

An important question arises now on a possible difference between the angular radius of the sun at one astronomical unit defined by $\sin\theta = R_\odot/a$ where $a = 1$au, and the radius of the disk $\rho = \sqrt{X^2 + Y^2}$ seen projected on the tangent plane. It is easy to see from the projection equations that $\rho = \tan\theta$. Then by neglecting the third order terms one can make $\rho = \theta$. The error is less than $0''.02$. More generally with Eqs. 1-3 one as $\tan\psi = \sqrt{X^2 + Y^2}$.

In this paper I have adopted the following reference values listed in Table 1 for all the numerical computations (geocentric or topocentric).

Table 1: *Standard values used for the numerical evaluations*

Parameter	Notation	Value	Unit
Angular solar radius at 1 AU	θ	959.50	$''$
Solar radius	$R_\odot$	695900	km
Astronomical unit	AU	149597870.61	km
Equatorial radius of the Earth	a_e	6378.136	km
Radius of Venus	R_p	6051.8	km
Solar parallax at one AU	$\pi_\odot$	8.794142	$''$
$TT - TU$ in June 2004	ΔT	64.6	s

3 Topocentric contacts

3.1 Apparent topocentric positions

An observer is located somewhere on the surface of the Earth with longitude and latitude λ and ϕ where the longitude is taken positive to the east of Greenwich. When a geocentric contact takes place, the apparent position of both Venus and the Sun are displaced from their geocentric position because of the parallactic angle between the centre of the Earth and its surface. The order of magnitude is $\sim 9''$ for the Sun and at most $36''$ for Venus. The maximum relative displacement is then $28''$. Therefore, given the typical relative velocity of Venus during a transit of 1.6 deg/day, the difference between the instants of

the contact computed at the centre of the Earth and those observable at the surface could be as large as ~ 400 seconds of time, and twice that value for the observations carried out at two nearly antipodal points on the Earth. It is time to note that the spheroidal shape of the Earth is marginally significant to determine the contact to the nearest second. The flattening being 1/298, in the most extreme case using a spherical Earth instead of the ellipsoid may cause an error just above one second of time. This will be neglected in the following, although this is a very easy to take it into account (the correct expression is given in Eqs. 72-75.

Table 2: *Useful notations*

Notation	Meaning
$\mathbf{r}_\odot$	Geocentric position vector of the Sun
$\mathbf{r}'_\odot$	Topocentric position vector of the Sun
$r_\odot$	Geocentric distance to the Sun
$r'_\odot$	Topocentric distance to the Sun
$\rho_\odot$	Geocentric angular radius of the Sun
$\mathbf{s}$	Geocentric unit vector to the Sun
$\mathbf{s}'$	Topocentric unit vector to the Sun
$\mathbf{r}_p$	Geocentric position vector of the planet
$\mathbf{r}'_p$	Topocentric position vector of the planet
r_p	Geocentric distance to the planet
r'_p	Topocentric distance to the planet
ρ_p	Geocentric angular radius of the planet
$\mathbf{p}$	Geocentric unit vector to the planet
$\mathbf{p}'$	Topocentric unit vector to the planet
$\mathbf{R}$	Geocentric radius vector of the observer
R	Distance of the observer to the Earth's centre ($\sim a_e$)
$\mathbf{u}$	Geocentric unit vector to the observer

Let $\mathbf{s}$ and $\mathbf{p}$ be respectively the geocentric unit vectors to the sun and the planet at the time of any of the geocentric contact and $\mathbf{r}_\odot$, $\mathbf{r}_p$ their position vectors. One notes also $r_\odot$ the geocentric distance of the Sun and r_p that of the planet. The cartesian coordinates of the planet with respect to the centre of the Sun on the tangent plane are always noted X, Y. Similar notations are used for the topocentric phenomena as $\mathbf{s}'$, $\mathbf{p}'$, $\mathbf{r}'_\odot$, $\mathbf{r}'_p$, $r'_\odot$, r'_p, X', Y'.

The vector position of the observer referred to the centre of the Earth is given by $\mathbf{R}$ and its cartesian coordinates with the origin of the longitude at the Greenwich meridian are,

$$x = a_e \cos\phi \, \cos\lambda \tag{8}$$

$$y = a_e \cos\phi \, \sin\lambda \tag{9}$$

$$z = a_e \sin\phi \tag{10}$$

where a_e could be replaced by a more accurate distance to allow for the Earth flattening. The geocentric unit vector in the direction of the observer is $\mathbf{R}/a_e = \mathbf{u}$.

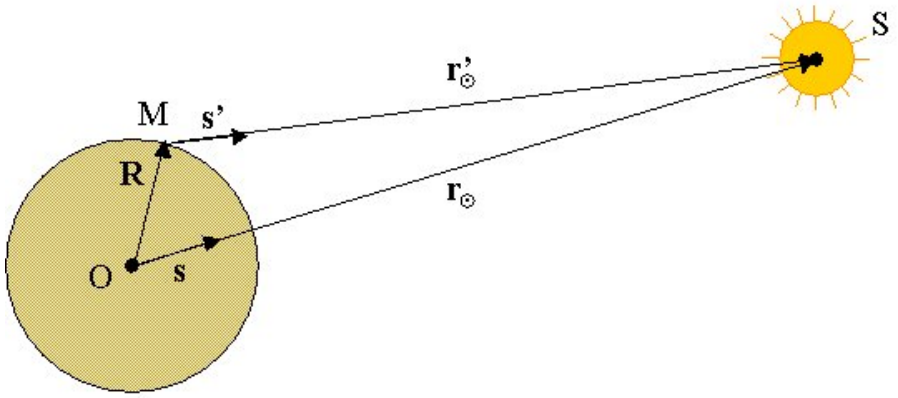


Figure 2: *Parallactic effect on the apparent direction of the Sun between the centre of the Earth* ($\mathbf{s}$) *and a point on its surface* ($\mathbf{s}'$). *A similar effect (albeit larger) applies also to Venus.*

With simple geometry one has without approximation,

$$\mathbf{r}'_\odot = \mathbf{r}_\odot - \mathbf{R} \tag{11}$$

and dividing both members by the geocentric solar distance $r_\odot$ this gives,

$$\frac{\mathbf{r}'_\odot}{r_\odot} = \mathbf{s} - \frac{\mathbf{R}}{r_\odot} \tag{12}$$

To express the correspondence between the geocentric and topocentric unit vectors one needs to evaluate the modulus of the left-hand side of Eq. 12. Squaring both members this yields,

$$\left|\frac{\mathbf{r}'_\odot}{r_\odot}\right| = \left(1 - 2\frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot} + \frac{R^2}{r_\odot^2}\right)^{1/2} \tag{13}$$

The second term in the above expression is of the order of the solar parallax($\sim 9'' \sim 10^{-4}$) whereas the last one is of the order of its square and can be neglected. This applies also to Venus. So to the first order in the parallax one has,

$$\left|\frac{\mathbf{r}'_\odot}{r_\odot}\right| \simeq 1 - \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot} \tag{14}$$

Then dividing (12) by (14) and discarding factors containing the square of parallax, one gets the relationship between the geocentric and topocentric unit vectors as,

$$\mathbf{s}' = \mathbf{s} - \frac{\mathbf{R}}{r_\odot} + \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot}\,\mathbf{s} \tag{15}$$

The same procedure can now be applied to the position of Venus at the same time which leads to,

$$\mathbf{p}' = \mathbf{p} - \frac{\mathbf{R}}{r_p} + \frac{\mathbf{p}\cdot\mathbf{R}}{r_p}\,\mathbf{p} \tag{16}$$

Finally the topocentric Sun-Venus vector is easily expressed as,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \mathbf{R}\left[\frac{1}{r_p} - \frac{1}{r_\odot}\right] + \frac{\mathbf{p}\cdot\mathbf{R}}{r_p}\,\mathbf{p} - \frac{\mathbf{s}\cdot\mathbf{R}}{r_\odot}\,\mathbf{s} \tag{17}$$

where one sees clearly the effect of the parallax. Qualitatively the situation is sketched out in Fig. 3, where the geocentric transit is shown at the instant of the second contact (left) and third contact (right). At the same times the topocentric position of Venus relative to that of the Sun is also shown in the same diagram. In this case the topocentric first inner contact happens at a later time compared to its geocentric counterpart. This is just the opposite for the third contact which comes earlier for the topocentric observer than at the geocenter. In general the total duration of the phenomena will differ from place to place.

If we now express the distances $r_\odot$ and r_p in astronomical units, considering a spherical Earth and by introducing the equatorial parallax of the Sun in Eq. 17 one has,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot\left[\mathbf{u}\left(\frac{1}{r_p} - \frac{1}{r_\odot}\right) + \frac{\mathbf{s}\cdot\mathbf{u}}{r_\odot}\,\mathbf{s} - \frac{\mathbf{p}\cdot\mathbf{u}}{r_p}\,\mathbf{p}\right] \tag{18}$$

Eq. 18 is nothing but a vectorial presentation of the relative diurnal parallax between two solar system bodies. It is convenient for the following to rewrite it as,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot\left[\left(\frac{1}{r_p} - \frac{1}{r_\odot}\right)\,(\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s}) + \frac{(\mathbf{s}\cdot\mathbf{u})\,\mathbf{s} - (\mathbf{p}\cdot\mathbf{u})\mathbf{p}}{r_p}\right] \tag{19}$$

where the last term in the bracket is of the order of $|\mathbf{s}-\mathbf{p}| \sim$ the solar radius and therefore about 200 times smaller than the main term.

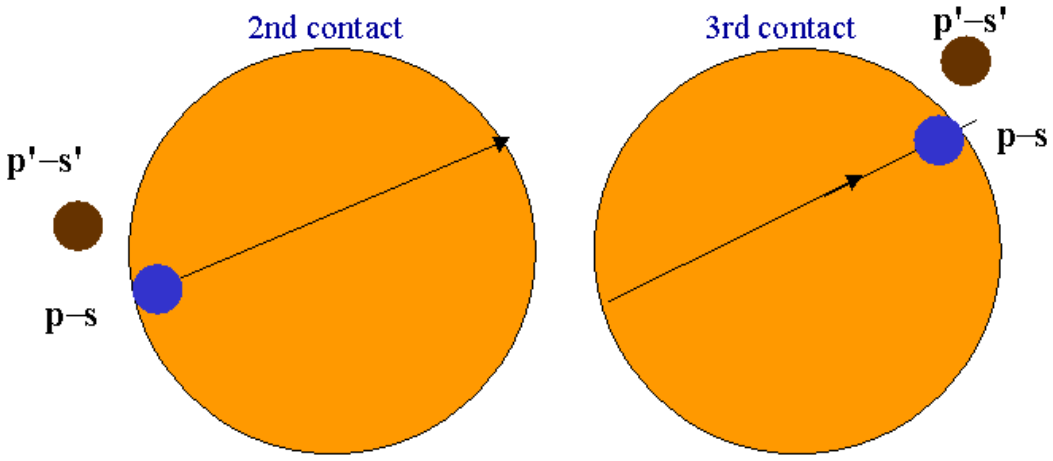


Figure 3: *Schematic diagram showing the geocentric and topocentric position of Venus relative to the centre of the Sun at the geocentric 2nd contact (left) or third contact (right). From the difference of position it is easy to evaluate the time difference between the events seen from different locations on the Earth and relate this difference to the solar parallax.*

3.2 Time gap between the topocentric and geocentric phenomena

It remains now to determine the time shift between the geocentric contacts and the same contacts observed at a particular location on the Earth surface. When Eq. 18 is evaluated at the instant of one of the geocentric contact, one knows that the angular shift of Venus from the topocentric contact is of the order of the relative parallax between Venus and the Sun, that is to say at most $30''$. So one needs only to evaluate the radial displacement of Venus with respect to the centre of the Sun to transform this distance into time needed to reach the contact, using in a first approximation the radial velocity of the geocentric phenomena.

The geocentric outer and inner contacts are characterised by the fact that,

$$|\mathbf{p}-\mathbf{s}| = \rho_\odot + \rho_p \quad (20)$$

$$|\mathbf{p}-\mathbf{s}| = \rho_\odot - \rho_p \quad (21)$$

expressing that the disk of Venus is tangent to that of the Sun on either side of the solar edge. For the transit of Venus of June 2004 one has typically $\rho_\odot = 945''$ and $\rho_p = 29''$.

If one has two vectors $\mathbf{v}$ and $\mathbf{v}'$ such that $\mathbf{v}' = \mathbf{v} + \mathbf{w}$ and $|\mathbf{w}| \ll |\mathbf{v}|$, then to the first order in $\mathbf{w}$ one has,

$$|\mathbf{v}'| \simeq |\mathbf{v}| + \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|} \quad (22)$$

or said differently, the difference of modulus is just the projection of the correction vector to the unit initial vector.

Applying Eq. 22 to Eq. 18 one finds readily,

$$\left|\mathbf{p}'-\mathbf{s}'\right| = |\mathbf{p}-\mathbf{s}| - \frac{\pi_\odot}{|\mathbf{p}-\mathbf{s}|}\left[\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})\left(\frac{1}{r_p}-\frac{1}{r_\odot}\right)+\frac{(\mathbf{s}\cdot\mathbf{u})\,(\mathbf{s}\cdot(\mathbf{p}-\mathbf{s}))}{r_\odot}-\frac{(\mathbf{p}\cdot\mathbf{u})\,(\mathbf{p}\cdot(\mathbf{p}-\mathbf{s}))}{r_p}\right] \tag{23}$$

This equation is in fact much simpler than one can imagine at first glance. The first term in the bracket is like $|\mathbf{p}-\mathbf{s}|$, that is to say to $\sin\theta$, the angular radius of the Sun. Consider the last two terms in the bracket. One has, $\mathbf{p}\cdot(\mathbf{p}-\mathbf{s}) = 1-\mathbf{p}\cdot\mathbf{s}$ and $\mathbf{s}\cdot(\mathbf{p}-\mathbf{s}) = -(1-\mathbf{p}\cdot\mathbf{s})$. With $\mathbf{p}\cdot\mathbf{s} = \cos(\theta)$, the factors are like $1-\cos\theta \sim \theta^2$ and about 200 times smaller than the main term. Then one can take $\mathbf{p}\cdot\mathbf{u} \sim \mathbf{s}\cdot\mathbf{u}$, meaning that one identifies the zenith distance of Venus to that of the centre of the Sun. Finally we can rewrite Eq. 23 as,

$$\left|\mathbf{p}'-\mathbf{s}'\right| = |\mathbf{p}-\mathbf{s}| - \frac{\pi_\odot}{|\mathbf{p}-\mathbf{s}|}\left[\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})\left(\frac{1}{r_p}-\frac{1}{r_\odot}\right)-(\mathbf{s}\cdot\mathbf{u})\,(1-\mathbf{p}\cdot\mathbf{s})\left(\frac{1}{r_p}+\frac{1}{r_\odot}\right)\right] \tag{24}$$

The first term in the bracket will produce time shifts as large as 400 s while the second will never add corrections larger than 2 seconds of time. But as both are linear in the solar parallax and in the position vector of the observer, there is no reason not to include them simultaneously.

The last step is now to compute the radial velocity of Venus on the Solar disk at each contact. Let $\mathbf{w}$ the transit velocity, that is to say the vector of components $\dot{X}$, $\dot{Y}$ on the projection plane. These components are computed at each geocentric contact and are numerical values for a particular transit. The radial velocity is then ,

$$w_r = \frac{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})}{|\mathbf{p}-\mathbf{s}|} \tag{25}$$

or

$$w_r = \frac{X\dot{X}+Y\dot{Y}}{\rho_\odot \pm \rho_p} \tag{26}$$

where the $\pm$ refers to the external or internal contacts. Combining 24 and 25 one eventually obtains the time of the topocentric contacts from the corresponding geocentric timings.

$$t'_i = t_i + \tau_i^{(1)} + \tau_i^{(2)} \tag{27}$$

with $i = 1..4$, where $\tau^{(1)}$ and $\tau^{(2)}$ denotes respectively the main and secondary terms from (24).

One gets,

$$\tau^{(1)} = \pi_\odot \left[\frac{\mathbf{u}\cdot(\mathbf{p}-\mathbf{s})}{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})} \left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) \right] \tag{28}$$

and

$$\tau^{(2)} = -\pi_\odot \left[\frac{(\mathbf{u}\cdot\mathbf{s})(1-\mathbf{p}\cdot\mathbf{s})}{\mathbf{w}\cdot(\mathbf{p}-\mathbf{s})} \left(\frac{1}{r_p} + \frac{1}{r_\odot} \right) \right] \tag{29}$$

Remarks about the units : The derivation assumes that all the angles, including the solar parallax, are in radians and the velocity in radian per unit of time. Care must be taken in the numerical evaluation mixing radians, degrees and arcseconds to ensure the homogeneity of the angular terms in the numerator and denominator. For example the magnitude of (28) is essentially $\pi_\odot/|\mathbf{w}|(1/r_p - 1/r_\odot)$. With $|\mathbf{w}| \sim 1.6$ deg/day, one must express $\pi_\odot$ in degrees, to obtain $\tau^{(1)} \sim 400$ s.

If we limit the evaluation to $\tau^{(1)}$ we can draw interesting conclusions from simple arguments :

- For a particular contact, all the vectors but $\mathbf{u}$ of Eq. 28 are known and do not depend on the observer. Putting for the unit vector $\mathbf{u}$,

$$\begin{aligned} \alpha &= \cos\phi\,\cos\lambda \\ \beta &= \cos\phi\,\sin\lambda \\ \gamma &= \sin\phi \end{aligned} \tag{30}$$

  we see that once reduced in numbers, assuming a known solar parallax, $\tau^{(1)}$ will reduce to

$$\boxed{A\alpha + B\beta + C\gamma} \tag{31}$$

  where A, B, C will depend only on the contact. In addition these coefficients are directly proportional to the solar parallax.

- From (31) or directly from the scalar product in (28) one sees that $\tau^{(1)} = \Gamma \times \cos\psi$ where ψ is the angle between $\mathbf{u}$ and an axis parallel to $\mathbf{p}-\mathbf{s}$ and going through the centre of the Earth with $\Gamma = (A^2 + B^2 + C^2)^{1/2}$. Therefore the difference between the timing of the geocentric contact and the topocentric timings goes from $+\Gamma$ to -Γ and is zero for observers located in a plane normal to $\mathbf{p}-\mathbf{s}$.

- The difference of the duration of the phenomena (topocentric - geocentric) between the second and the third contact will be given by,

$$(A_3 - A_2)\alpha + (B_3 - B_2)\beta + (C_3 - C_2)\gamma \tag{32}$$

having then the same form as for either contacts. So one can also define an axis on the Earth so that $\tau_3 - \tau_2 = \Gamma \cos\psi$. This remarkable property will be essential to assess the solar parallax from observations performed at two or more locations.

3.3 Practical formulas

The various scalar products can be evaluated in the most convenient frame. The position of the observer is naturally known in a frame tied to the rotating Earth with the origin of longitude at the Greenwich meridian. The components of the vector $\mathbf{p} - \mathbf{s}$ are known in cartesian coordinates in the plane tangent to the celestial sphere in the direction of the Sun. The coordinates X, Y are given for each contact by Eqs. 1 or 4.

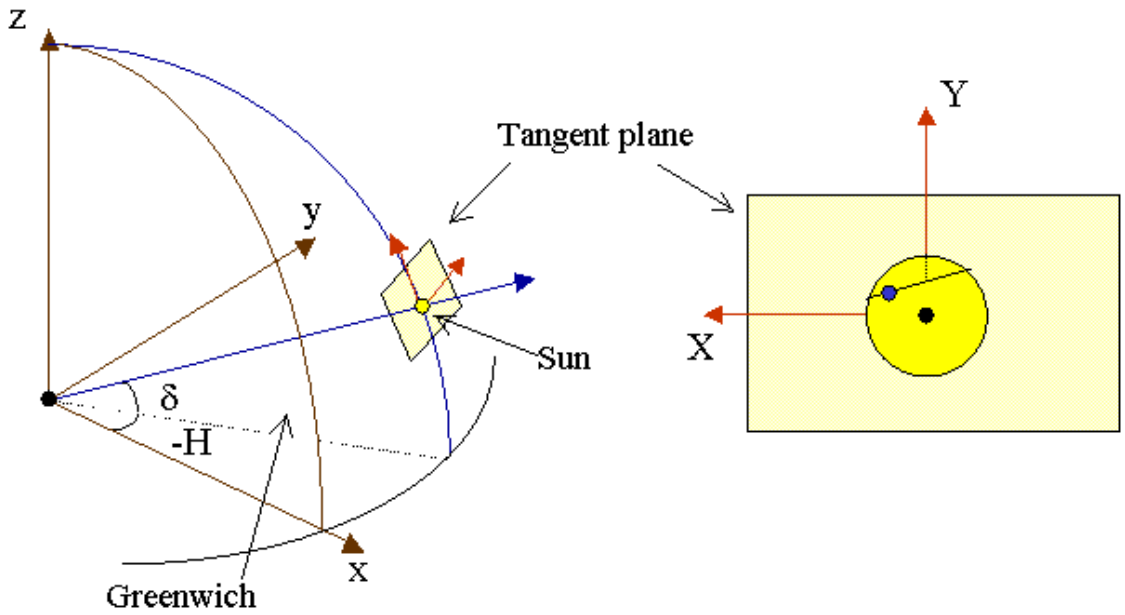


Figure 4: *Relationship between the local reference frame on the tangent plane centered at the Sun and the hour system with the Greenwich meridian for the origin of the longitudes and the hour angles. The close-up on the right is viewed from inside the celestial sphere.*

In the local frame attached to the centre of the Sun on has,

$$\mathbf{p} - \mathbf{s} = \begin{bmatrix} 0 \\ X \\ Y \end{bmatrix} \tag{33}$$

Let $H_\odot$ be the hour angle of the Sun for the Greenwich meridian at one of the contacts (omitting the subscript i to label the contact) and $\delta_\odot$ its declination. Therefore the matrix

that transforms the components of a vector from the local tangent frame into the hour system is given by the application of two rotations,

$$\mathcal{R}_3(H_\odot)\mathcal{R}_2(\delta_\odot) = \begin{bmatrix} \cos H_\odot \cos\delta_\odot & \sin H_\odot & -\cos H_\odot \sin\delta_\odot \\ -\sin H_\odot \cos\delta_\odot & \cos H_\odot & \sin H_\odot \sin\delta_\odot \\ \sin\delta_\odot & 0 & \cos\delta_\odot \end{bmatrix} \tag{34}$$

Thus one has in the hour frame,

$$\mathbf{p}-\mathbf{s} = \begin{bmatrix} X\ \sin H_\odot - Y\ \cos H_\odot \sin\delta_\odot \\ X\ \cos H_\odot + Y\ \sin H_\odot \sin\delta_\odot \\ Y\ \cos\delta_\odot \end{bmatrix} \tag{35}$$

which leads to,

$$\tau^{(1)} = A\ \cos\phi\cos\lambda + B\ \cos\phi\sin\lambda + C\ \sin\phi \tag{36}$$

with

$$A = \frac{\pi_\odot}{X\dot{X}+Y\dot{Y}}\left[\frac{1}{r_p}-\frac{1}{r_\odot}\right](X\ \sin H_\odot - Y\ \cos H_\odot \sin\delta_\odot) \tag{37}$$

$$B = \frac{\pi_\odot}{X\dot{X}+Y\dot{Y}}\left[\frac{1}{r_p}-\frac{1}{r_\odot}\right](X\ \cos H_\odot + Y\ \sin H_\odot \sin\delta_\odot) \tag{38}$$

$$C = \frac{\pi_\odot}{X\dot{X}+Y\dot{Y}}\left[\frac{1}{r_p}-\frac{1}{r_\odot}\right](Y\ \cos\delta_\odot) \tag{39}$$

We have also

$$\tau^{(2)} = \delta A\ \cos\phi\cos\lambda + \delta B\ \cos\phi\sin\lambda + \delta C\ \sin\phi \tag{40}$$

where the only vector to be expressed in the hour frame is $\mathbf{s}$,

$$\mathbf{s} = \begin{bmatrix} \cos\delta_\odot \cos H_\odot \\ -\cos\delta_\odot \sin H_\odot \\ \sin\delta_\odot \end{bmatrix} \tag{41}$$

giving

$$\delta A = -\frac{2\pi_\odot}{X\dot{X}+Y\dot{Y}}\ \sin^2\frac{\theta}{2}\ \left[\frac{1}{r_p}+\frac{1}{r_\odot}\right]\cos\delta_\odot\cos H_\odot \tag{42}$$

$$\delta B = \frac{2\pi_\odot}{X\dot{X}+Y\dot{Y}}\ \sin^2\frac{\theta}{2}\ \left[\frac{1}{r_p}+\frac{1}{r_\odot}\right]\cos\delta_\odot\sin H_\odot \tag{43}$$

$$\delta C = -\frac{2\pi_\odot}{X\dot{X}+Y\dot{Y}}\ \sin^2\frac{\theta}{2}\ \left[\frac{1}{r_p}+\frac{1}{r_\odot}\right]\sin\delta_\odot \tag{44}$$

As mentioned earlier the coefficients A, B, C are of the order of few hundred seconds while the δA, δB, δC reach only one or two seconds of time. From comparisons with rigourous estimates of the topocentric phenomenons for various locations, I estimate the accuracy of the prediction of the topocentric contacts with the approximate formulas to better than 0.1 minute of time at most locations, with maximum departure of 10-15s at some places. This is sufficient to process observational material, given the difficulties to ascertain the instant of the contacts even within 10 seconds. For the best observations, more accurate expressions (but also more intricate) derived in the next sections may be needed.

4 Further refinements

4.1 Statement of the problem

It is shown later in this note that the accuracy of the linear formula for the instant of the topocentric contacts is in general better than 10 seconds, with a well defined pattern as a function of the location. It is also found that the errors for any of the four contacts do not average out over the earth when locations are picked uniformly distributed, a feature not accounted for by the linear formula. The relatively poor performances of the linear approximation is rather surprising and not intuitive, since the main approximation comes from neglecting the square of the parallax in Eq. 18. As the difference between the geocentric and topocentric contacts is at most $\Delta\tau \sim 500$s, one expects the neglect terms to be of the order of $\pi_\odot \, \Delta\tau \sim 0.05\,$s and not the 5–10 s eventually found.

This unaccounted discrepancy is both puzzling and challenging. It must be investigated, at least to understand its origin and possibly to produce a better formula. There are clearly second order terms missing, larger than simply those that would come directly from $\pi_\odot^2$ in the parallactic equation and unrelated to them. I eventually found the culprit hidden away in a dark corner.

If we look again at Eq. 18 one sees that the parallactic term is small compared to $\mathbf{p}-\mathbf{s}$ and not relative to the unit vectors $\mathbf{p}$ or $\mathbf{s}$, as it was the case in Eqs. 15-16. As $|\mathbf{p}-\mathbf{s}| \sim \theta_\odot$, the second order terms is typically $\pi_\odot/\theta_\odot \sim 0.01$ smaller than the main term and not simply $\pi_\odot \sim 10^{-4}$. This means that some second order terms in the derivation of $\Delta\tau$ will be of the order of ~ 5 s, instead of 0.05 s, and this will happen even with the parallactic equation limited to first order in $\pi_\odot$. This difference of few seconds is precisely the accuracy of the linear formula found from a comparison to a full numerical computation of the contacts. One then needs to recompute the topocentric timings by allowing for these second order terms.

4.2 Second order equations

As explained there is no need at this point to introduce the square of the parallax in the basic equations (18) or (19). The problem can then be restated as follows : given the topocentric position of Venus relative to the Sun as,

$$\mathbf{p}' - \mathbf{s}' = \mathbf{p} - \mathbf{s} - \pi_\odot \left[\left(\frac{1}{r_p} - \frac{1}{r_\odot} \right) (\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s}) + \frac{(\mathbf{s}\cdot\mathbf{u})\,\mathbf{s} - (\mathbf{p}\cdot\mathbf{u})\mathbf{p}}{r_p} \right] \tag{45}$$

and knowing the geocentric motion of the two bodies and that of the observer, one wishes to solve for the time t the equations,

$$|\mathbf{p}'(t) - \mathbf{s}'(t)| = \rho_\odot \pm \rho_p \tag{46}$$

The solutions of (46) will be the instant of the four topocentric contacts and it must be obtained with a better accuracy than that of the linear model. We also assume that the solution for the geocentric contacts is known, meaning that at for certain t_0,

$$|\mathbf{p}(t_0) - \mathbf{s}(t_0)| = \rho_\odot \pm \rho_p \tag{47}$$

has been solved. To avoid too many subscripts, the contacts are not indexed here and the formulas are meant to apply to any of the four contacts. For brevity one puts hereafter $\mathbf{v} = \mathbf{p} - \mathbf{s}$ and $\mathbf{v}' = \mathbf{p}' - \mathbf{s}'$ and introduce $1/p = 1/r_p - 1/r_\odot$ for the parallactic factor. In this discussion one can restrict the parallactic equation (45) to its simplest form,

$$\mathbf{v}' = \mathbf{v} - \frac{\pi_\odot}{p}\,[\mathbf{u} - (\mathbf{s}\cdot\mathbf{u})\mathbf{s}] \tag{48}$$

since we have already shown that the additional terms give corrections of the order of ~ 1 s (29) and that they will not interfere with the second order terms we are about to derive.

Let t_0 the time of a geocentric contact and ρ the distance to the center of the Sun (so ρ is either $\rho_\odot + \rho_p$ or $\rho_\odot - \rho_p$ according to the contact). At this time

$$\mathbf{v}^2(t_0) = \rho^2 \tag{49}$$

(the equations are easier to derive with the square of the distance). The same contact for an Earth-based observer occurs at t with

$$\mathbf{v}'^2(t) = \rho^2 \tag{50}$$

Now with (48)

$$\mathbf{v}'^2 = \mathbf{v}^2 - 2\frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{v} + \frac{\pi_\odot^2}{p^2}\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right] \tag{51}$$

and by inserting the geocentric contact, the equation for the time of the topocentric contact reduces to,

$$\boxed{\mathbf{v}^2(t) - \mathbf{v}^2(t_0) - 2\frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{v} + \frac{\pi_\odot^2}{p^2}\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right] = 0} \tag{52}$$

This equation is very general and the only approximation results from our neglecting $\pi_\odot^2$ in the initial parallactic equation. As mentioned earlier this does not conflict with the second order terms of (52), since the magnitude of the second order term relative to the first order is $\pi_\odot/|\mathbf{v}| \sim 0.01$ and not $\pi_\odot \sim 10^{-4}$.

In the vicinity of the contact the geocentric motion of Venus relative to the Sun is nicely approximated by a rectilinear uniform motion as,

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \mathbf{w}\,\tau \tag{53}$$

and the motion of the observer is provided by the rotation of the Earth as,

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \boldsymbol{\omega}\times\mathbf{u}(t_0)\,\tau \tag{54}$$

where $\boldsymbol{\omega}$ is the rotation vector of the Earth and $\tau = t - t_0$. Inserting in (52) this leads to,

$$\frac{\pi_\odot}{p}\,\mathbf{u}\cdot\mathbf{v} - \frac{1}{2}\frac{\pi_\odot^2}{p^2}\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right] - \left[\mathbf{v}\cdot\mathbf{w} - \frac{\pi_\odot}{p}(\boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{v} - \frac{\pi_\odot}{p}\mathbf{u}\cdot\mathbf{w}\right]\tau - \frac{1}{2}\mathbf{w}^2\tau^2 = 0 \tag{55}$$

Although this quadratic equation in τ can be solved in closed form it is better to use the fact that $\tau \sim \pi_\odot/|\mathbf{w}|$ to solve the linear part first and add the quadratic term as a correction. After few algebraic transformations one gets,

$$\begin{aligned}\tau \;=\;& \frac{\pi_\odot}{p}\,\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{w}\cdot\mathbf{v}} \\ &+ \frac{\pi_\odot^2}{p^2}\left[\frac{(\mathbf{u}\cdot\mathbf{v})((\boldsymbol{\omega}\times\mathbf{u})\cdot\mathbf{v})}{(\mathbf{w}\cdot\mathbf{v})^2} + \frac{(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})}{(\mathbf{w}\cdot\mathbf{v})^2} - \frac{1}{2}\frac{\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right]}{\mathbf{w}\cdot\mathbf{v}} - \frac{1}{2}\,\frac{(\mathbf{w}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})^2}{(\mathbf{w}\cdot\mathbf{v})^3}\right]\end{aligned} \tag{56}$$

where the first order term is the same as (28). The complementary expression (29) does not appear as expected since I have discarded the specific small terms in the parallactic equation (48). So Eq. 29,

$$\tau^{(2)} = -\pi_\odot \left[\frac{(\mathbf{u}\cdot\mathbf{s})(1-\mathbf{p}\cdot\mathbf{s})}{\mathbf{w}\cdot\mathbf{v}} \left(\frac{1}{r_p} + \frac{1}{r_\odot} \right) \right] \tag{57}$$

should be added to this result to get the correct timing to the second order.

It is interesting to rewrite (56) by introducing,

$$\mathbf{\Gamma} = (A, B, C)^T = \frac{\pi_\odot}{p} \frac{\mathbf{v}}{\mathbf{w}\cdot\mathbf{v}} \tag{58}$$

where the components A, B, C are the same as in (37). Put also,

$$\mathbf{\Gamma'} = (A', B', C')^T = \frac{\pi_\odot}{p} \frac{\mathbf{w}}{\mathbf{w}\cdot\mathbf{v}} \tag{59}$$

Then (56) becomes,

$$\begin{aligned} \tau \;=\; & \mathbf{\Gamma}\cdot\mathbf{u} + (\mathbf{\Gamma}\cdot\mathbf{u})\left[\mathbf{\Gamma}\cdot(\boldsymbol{\omega}\times\mathbf{u})\right] + (\mathbf{\Gamma}\cdot\mathbf{u})\,(\mathbf{\Gamma'}\cdot\mathbf{u}) \\ & -\frac{1}{2}\frac{\pi_\odot^2}{p^2}\frac{\left[1-(\mathbf{s}\cdot\mathbf{u})^2\right]}{\mathbf{w}\cdot\mathbf{v}} - \frac{1}{2}\frac{(\mathbf{w}\cdot\mathbf{w})}{(\mathbf{w}\cdot\mathbf{v})}(\mathbf{\Gamma}\cdot\mathbf{u})^2 \end{aligned} \tag{60}$$

With this last equation the question raised at the beginning of this section is answered, since we have now a solution for the time of the contacts including second order terms. A quick evaluation of each term with $\mathbf{u}^2$ indicates that they are all of the order of 1 to 5 s, as expected.

4.3 Convenient representation

It appears from these equations that the time difference between the geocentric and topocentric contacts can be expressed as a function of the location of the observer with an expansion involving only first and second degree of $\mathbf{u}$. A full expansion of (56) will lead to a second degree polynomial in α, β, γ. So it is possible to derive a certain number of constants, depending only on the transit and the contacts but not on the observer, from which any timing could be computed for a particular location. We have already done this expansion with the linear term in $\mathbf{u}$ and it is wise to organise the expansion to the second order on the spherical functions of the first and second degree. Our previous expansion turns out to be identical to such a development on the spherical function of first order since,

$$\begin{aligned} C_{11} &= \cos\phi\cos\lambda &&= \alpha \\ S_{11} &= \cos\phi\sin\lambda &&= \beta \\ C_{10} &= \sin\phi &&= \gamma \end{aligned} \tag{61}$$

Table 3: *Coefficients of the expansion of (60) on the spherical functions.*

Term	$\mathbf{\Gamma}\cdot\mathbf{u}$	$(\mathbf{\Gamma}\cdot\mathbf{u})\,[\mathbf{\Gamma}\cdot(\boldsymbol{\omega}\times\mathbf{u})]$	$(\mathbf{\Gamma}\cdot\mathbf{u})\,(\mathbf{\Gamma}'\cdot\mathbf{u})$	$(\mathbf{\Gamma}\cdot\mathbf{u})^2$	$1-(\mathbf{s}\cdot\mathbf{u})^2$
C_{00}	0	0	$(AA'+BB'+CC')/3$	$(A^2+B^2+C^2)/3$	$2/3$
C_{11}	A	0	0	0	0
S_{11}	B	0	0	0	0
C_{10}	C	0	0	0	0
C_{22}	0	$AB/3$	$(AA'-BB')/6$	$(A^2-B^2)6$	$(S_y^2-S_x^2)/6$
S_{22}	0	$(B^2-A^2)/6$	$(AB'+BA')/6$	$AB/3$	$-S_xS_y/3$
C_{21}	0	$BC/3$	$(AC'+CA')/6$	$2AC/3$	$-2S_xS_z/3$
S_{21}	0	$-AC/3$	$(BC'+CB')/6$	$2BC/3$	$-2S_yS_z/3$
C_{20}	0	0	$-(AA'+BB'-2CC')/3$	$-(A^2+B^2-2C^2)/3$	$(1-3S_z^2)/2$

Now more generally one has,

$$\begin{array}{rcll} C_{00} &=& 1 & \\ C_{22} &=& 3\cos^2\phi\,\cos 2\lambda &= 3(\alpha^2-\beta^2) \\ S_{22} &=& 3\cos^2\phi\,\sin 2\lambda &= 6\alpha\beta \\ C_{21} &=& 3\cos\phi\sin\phi\,\cos\lambda &= 3\alpha\gamma \\ S_{21} &=& 3\cos\phi\sin\phi\,\sin\lambda &= 3\beta\gamma \\ C_{20} &=& (3\sin^2\phi-1)/2 &= (3\gamma^2-1)/2 \end{array} \tag{62}$$

There are in total only nine base functions, since $\alpha^2+\beta^2+\gamma^2=1$. It remains to expand (60) on $\alpha,\cdots,\alpha^2,\cdots,\alpha\beta\cdots$ and then on the C_{lm}, S_{lm} to obtain a general formula applicable to any locations. For the user the nine numerical values for each contact can be computed in advance and provided in the form of table. The analytical form of the contribution of each second order factor of (60) is given in Table 3. One must note in this table that the first line gives the average of the difference between the topocentric and geocentric timing over the Earth. For the 2004 Venus transit the numerical values are listed in Tables 5 and 6.

Table 4: *Venus transit of 8 June 2004 – Geocentric contacts*

Contact	Time	X	Y	$\dot{X}$	$\dot{Y}$	θ	$H_\odot$
	UT	$''$	$''$	$''$/h	$''$/h	$''$	deg
1st	$05^h13^m34^s$	873.44	-431.34	-233.62	-57.35	945.40	258.62
2nd	$05^h32^m51^s$	798.41	-449.76	-233.62	-57.35	945.39	263.45
3rd	$11^h06^m41^s$	-501.99	-766.61	-233.80	-56.53	945.37	-13.10
4th	$11^h25^m58^s$	-577.09	-784.78	-233.80	-56.53	945.37	-8.28

5 Numerical Values

5.1 The geocentric transit

The circumstances of the Venus transit of June 2004 have been computed with the reference values listed in Table 1 and solar system ephemeris developed over the years at the Institut de Mécanique Céleste et de Calcul des Ephémérides in Paris. The timings are computed in dynamical times but given as usual in UT, using $TT - UT = 64.6$s in June 2004. This value may differ by few seconds from one institute to another and will be known for sure at the end of 2004. The predictions are also very sensitive to the solar radius : increasing this radius by 100 km, or equivalently by about $0''.1$ (this is less than the current uncertainty of the angular radius of the Sun), advances the first two contacts of 2 seconds and delays the last contacts by the same amount. So the length of the transit is increased of 4s. This is then close to the absolute accuracy reachable by any prediction, even though the numerical model of the motion of the Earth and Venus would allow to achieve something much better, by several orders of magnitude. The geocentric parameters of the transit are given in Table 4.

5.2 The topocentric transit

To obtain a good approximation of the timing at any point of the Earth and also the duration of the transit one needs only to compute the coefficients A, B, C (and A', B', C' for greater precision). The values are listed in Table 5, in cartesian and spherical coordinates, using respectively b and l for the latitude and longitude of the representative point on the

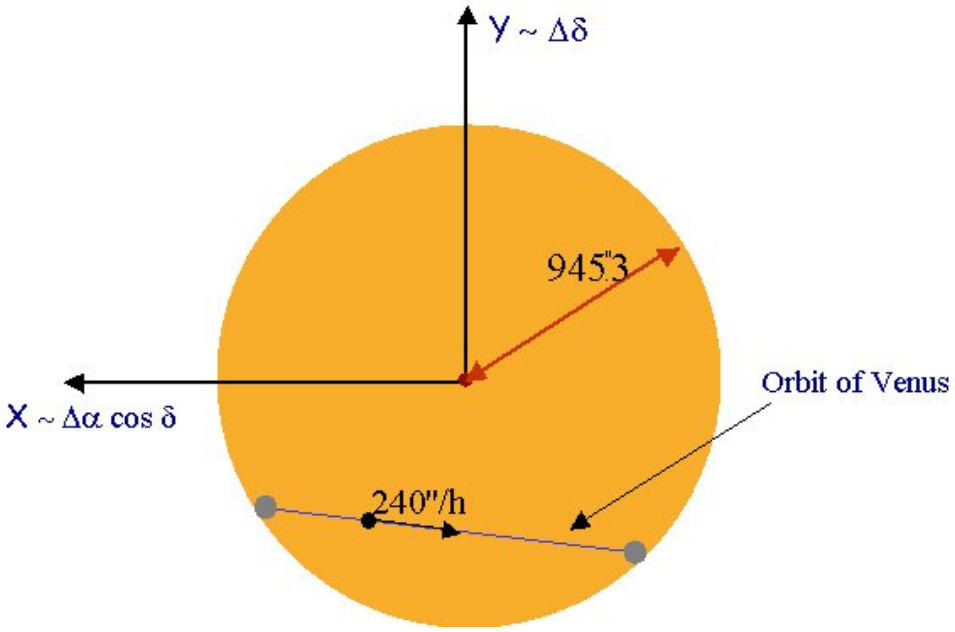


Figure 5: *Geometry of the passage of 8 June 2004.*

Earth surface, with,

$$A + \delta A = \Gamma \cos b \cos l \tag{63}$$

$$B + \delta B = \Gamma \cos b \sin l \tag{64}$$

$$C + \delta C = \Gamma \sin b \tag{65}$$

With the last two lines of Table 5 one can evaluate the duration of the transit between the two internal contacts from any place on the Earth. It remains to check that the Sun is above the horizon at this time. However the result of the computation is precisely what would be observed through a transparent Earth if the sun is set !. The accurate value of the difference between the topocentric and geocentric time of the contacts and the duration of the inner transit is plotted in Fig. 6 for any point on the Earth, letting aside the problem of visibility. The plots show obvious pattern of regularity with the maximum deviation around the poles indicated in Table 5. The model with 3 (linear) or 9 (quadratic) parameters aims precisely to account for this pattern.

5.3 Accuracy of the approximation

Earlier in this note I have estimated the accuracy of the simplified formula to 5–10 s, without providing hints for the magnitude of the neglected terms. This was based on quick comparison between predictions based on the linear model and a rigourous numerical

Table 5: *Venus transit of June 2004. Coefficients for the topocentric contacts : linear formula.*

Contact	$A+\delta A$ s	$B+\delta B$ s	$C+\delta C$ s	Γ s	b deg	l deg
1st	388.6	4.9	174.4	425.9	24.2	0.7
2nd	396.5	-38.6	202.9	447.0	27.0	354.4
3rd	195.5	-206.0	-345.3	447.1	-50.6	313.5
4th	166.9	-230.7	-316.8	426.0	-48.1	305.9
2nd to 3rd	-200.9	-167.4	-548.2	607.3	-64.5	219.8
1st to 4th	-221.6	-235.7	-491.2	588.2	-56.6	226.8

computation of the time of the contacts. About the accuracy one must already notice that the various official, and basically very trustworthy, computations carried out in Paris, Washington or by Nasa, may differ by few seconds in the geocentric phenomena. This comes primarily, not from the model as stated earlier, but from the adopted angular radius of the Sun, from the prediction of the rotation of the Earth in June 2004, and to a lesser extent to the ephemerides they use. Therefore a precision of one second, is not only not required for the observations, but is difficult to achieve in absolute sense with our knowledge of the solar diameter.

However, within a given set of astronomical constants, including a conventional size of the Sun, it is possible to compare the prediction of the linear model to an exact computation of the contact timings at different locations on the Earth. I have determined this precise timings for about 3000 locations on the Earth, regularly distributed on its surface, using a spherical Earth for the purpose of comparison with the model (the main difference would have appeared in the coefficient C which should be corrected by a factor $1-2f$ for the Earth flattening). For this check of a purely mathematical formula, the actual visibility of the phenomena is of no importance. Stated differently the computation is done as if the Earth were a transparent sphere with the Sun observable from any location (this situation would be perfectly relevant for the observation of a transit with neutrino detectors !). Then for each contact and for all the places the differences between the geocentric and topocentric phenomena have been fitted with a least squares model to the simplified and full model. This allows to check the accuracy of each model and at the same time to compare the numerical value of each of the coefficients given in Table 5 for the linear

Table 6: *Venus transit of June 2004. Coefficients for the topocentric contacts : second order terms.*

Contact	C_{00} s	C_{22} s	S_{22} s	C_{21} s	S_{21} s	C_{20} s
1st	2.7	0.0	-2.1	1.5	-3.2	3.8
2nd	3.3	-0.5	-2.2	1.5	-4.0	4.7
3rd	-3.3	-1.0	-0.3	3.6	3.5	-4.7
4th	-2.7	-1.0	0.0	3.4	2.7	-3.8
2nd to 3rd	-6.7	-0.5	1.9	2.1	7.5	-9.4
1st to 4th	-5.4	-1.0	2.1	1.9	5.9	-7.6

model and Table 6 for the second order model.

For each contact (or duration) i and place k one has an equation like

$$d\tau_k^i = A^i\alpha + B^i\beta + C^i\gamma \tag{66}$$

or

$$d\tau_k^i = \sum_{l,m} c_{lm}\, C_{lm}(\phi,\lambda) + s_{lm}\, S_{lm}(\phi,\lambda) \tag{67}$$

according to the model.

The results are shown in Table 7 (linear model) Table 8 (full model) for the second and third contact and for the duration of the transits between these contacts.

A straight comparison with Tables 5-6, confirms that the analytic modelling and the numerical evaluation of the coefficients are correct, and that not term of second order is missing. The remaining difference between the coefficients is less than 0.1 s. The last two columns give the average of the post-fit residuals and their standard deviation. As mentioned earlier, the linear model leaves residuals between 5 and 10 s, which do not average out over the Earth. The standard deviation is about 5 s for the contacts and 8s for the duration. Results are exactly similar for the first and fourth contact.

The distribution of the residuals is shown in Fig. 7 (linear model) and 8 (second order model) for the duration between the second and third contact. In the first case we see a

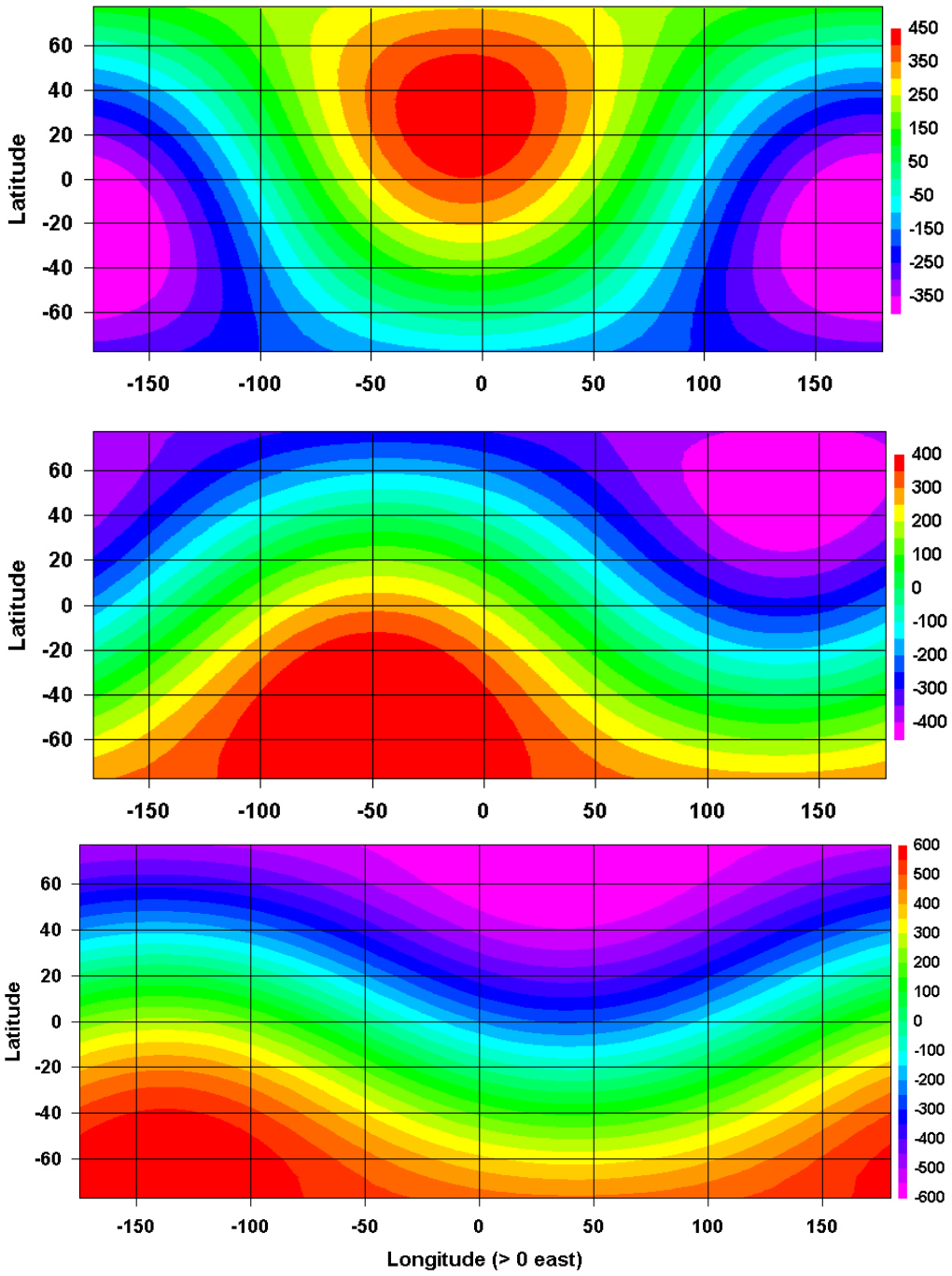


Figure 6: *Difference in seconds between the geocentric and topocentric time of the second (top) and third contact (middle), and difference of duration of the transit (bottom) for any point on the Earth (irrespective of the visibility). The plot is based on a rigorous computation for the contacts of the 2004 transit.*

Table 7: *Venus transit of June 2004 : Coefficients for the topocentric contacts fitted with the linear model onto an accurate determination of the contacts on uniformly sampled points on the Earth.*

Contact	$A + \delta A$ s	$B + \delta B$ s	$C + \delta C$ s	Mean s	σ s
2nd	396.4	-38.6	203.0	3.4	5.2
3rd	195.5	-205.8	-345.4	-3.3	4.7
2nd to 3rd	-200.9	-167.2	-548.5	-6.6	8.0

very distinctive pattern as a function of the longitude and latitude with a non-zero average over the surface of the Earth. The amplitudes around the mean have a maximum of 10 seconds for the contacts and 15s for the duration of the transit and the standard deviations are respectively 5 and 8 seconds, that is to say in keeping with the claim of an accuracy of 0.1 mn for the linear formula. When the full model is applied the residuals are down by almost two orders of magnitude, and the remaining effect averages out to zero although the remaining residuals are not yet pure noise. However there is neither theoretical nor practical need to derive a more accurate analytical formula than (56) for the timings.

Table 8: *Venus transit of June 2004 : Coefficients for the topocentric contacts fitted with the complete model onto an accurate determination of the contacts on uniformly sampled points on the Earth.(The coefficients of the linear model are not repeated)*

Contact	C_{00} s	C_{22} s	S_{22} s	C_{21} s	S_{21} s	C_{20} s	Mean s	σ s
2nd	3.4	-0.4	-2.2	1.5	-4.0	4.7	0	0.14
3rd	-3.3	-1.0	-0.3	3.6	3.5	-4.7	0	0.11
2nd to 3rd	-6.6	-0.5	1.9	2.1	7.5	-9.4	0	0.19

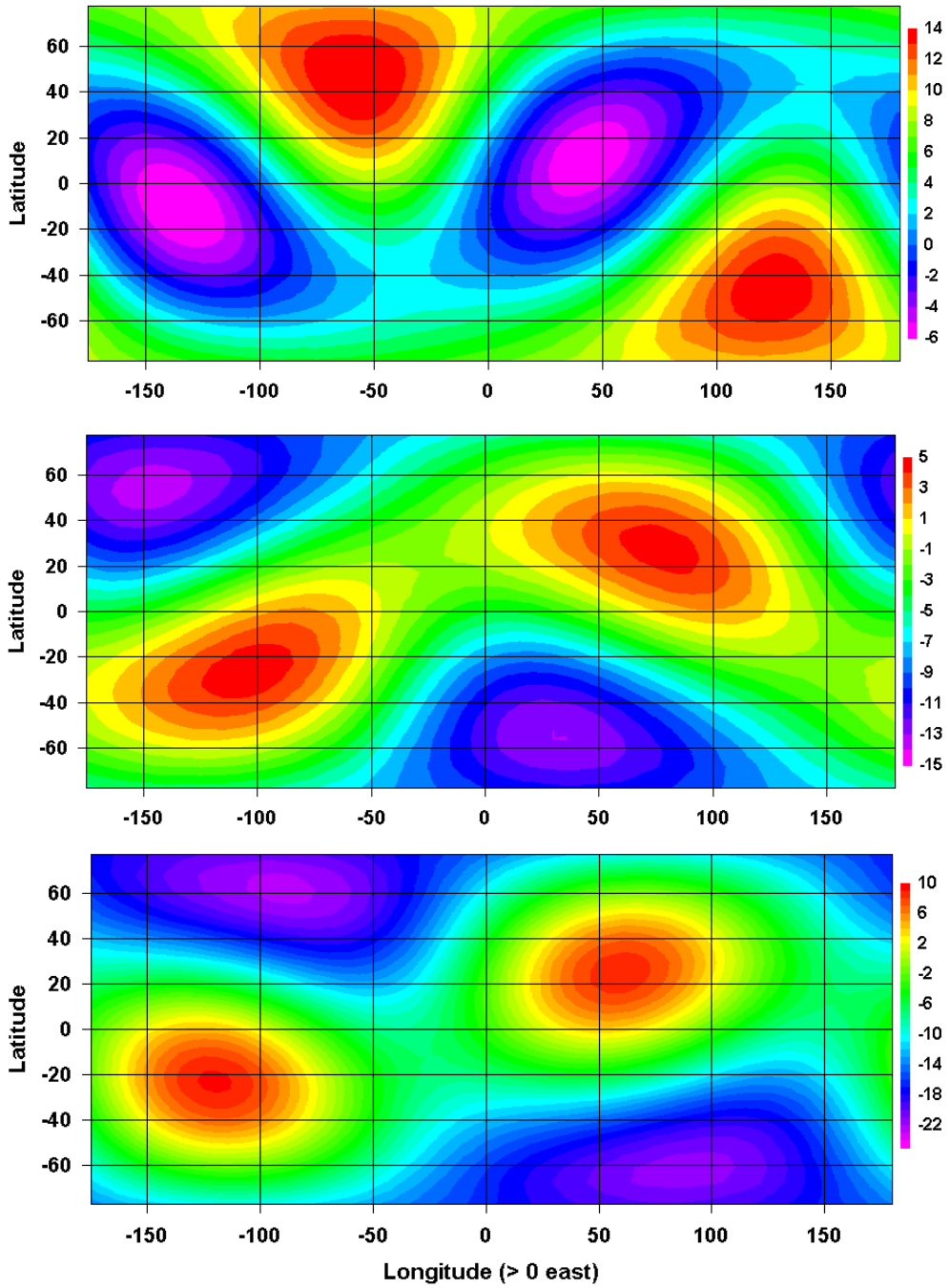


Figure 7: *Residuals in seconds for the 2004 Venus transit in the timing of the second (top) and third contact (middle) and the transit duration (bottom) between the exact computation and the linear model with three coefficients using the best fitted values of Table 7, for any point on the Earth (irrespective of the visibility). Note the amplitude of* ~ 10 *s and the non zero average.*

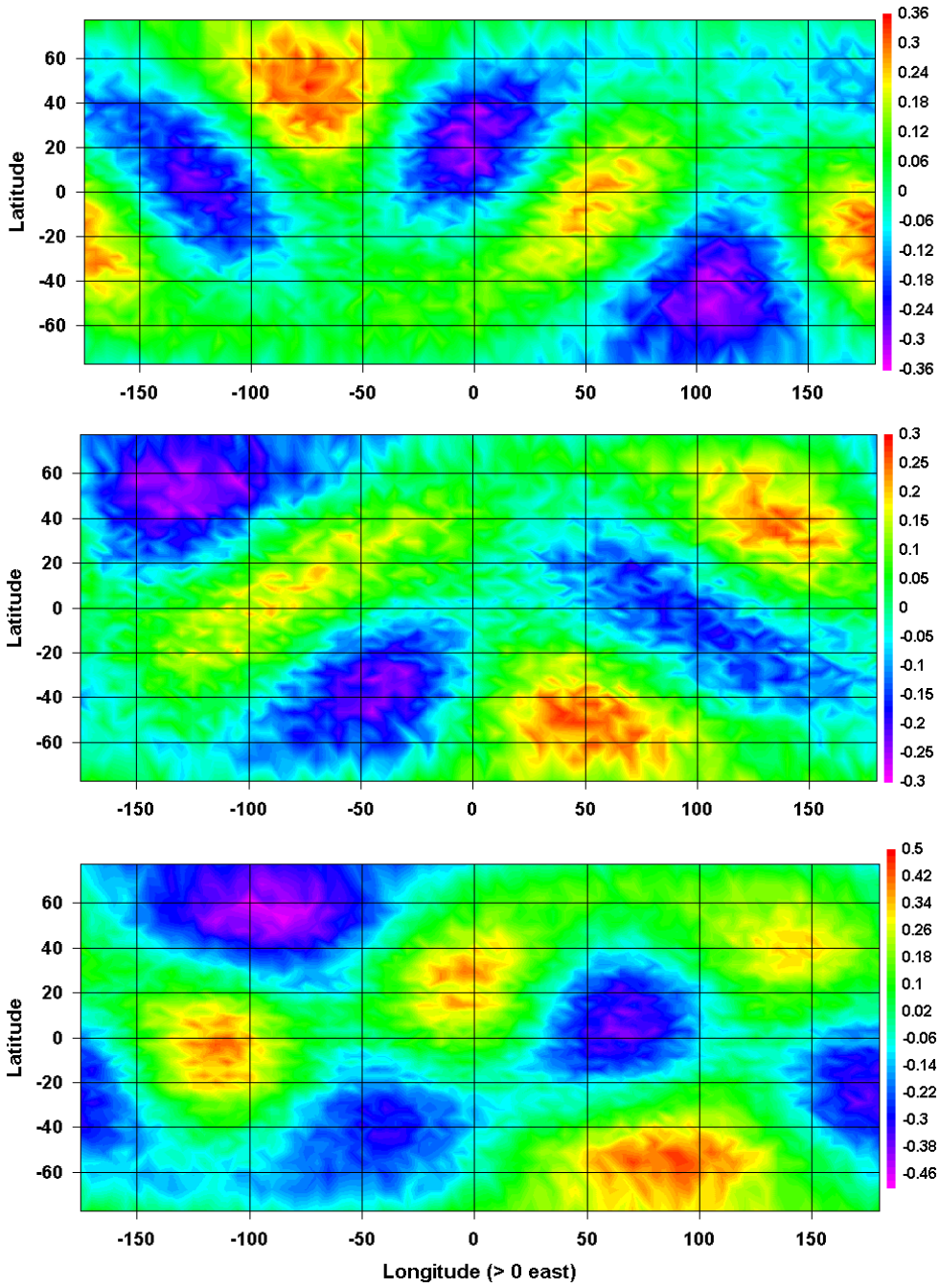


Figure 8: *Residuals in seconds for the 2004 Venus transit in the timing of the second (top) and third contact (middle) and the transit duration (bottom) and between the exact computation and the second order model with nine coefficients using the best fitted values of Table 8, for any point on the Earth. The amplitude is less than 0.5 s and the average is zero.*

6 Determination of the solar parallax

In this section an application of the linear model is proposed in view of processing timings to determine the astronomical unit. As explained in the previous sections the time difference between the geocentric and topocentric contacts is given by the simple expression,

$$\tau = A\alpha + B\beta + C\gamma \tag{68}$$

where $A, \cdots$ stands in fact for $A + \delta A, \cdots$ and α, β, γ are the direction cosines of the observer referred to the equator and the Greenwich meridian with,

$$\begin{aligned} \alpha &= \cos\phi \, \cos\lambda & (69)\\ \beta &= \cos\phi \, \sin\lambda & (70)\\ \gamma &= \sin\phi & (71) \end{aligned}$$

If one wishes to allow for the flattening of the Earth this must be replaced by,

$$\begin{aligned} \alpha &= \rho(\phi) \cos\phi \, \cos\lambda & (72)\\ \beta &= \rho(\phi) \cos\phi \, \sin\lambda & (73)\\ \gamma &= \rho(\phi)(1-2f) \sin\phi & (74) \end{aligned}$$

with,

$$\rho(\phi) = (1 - 2f \, \sin^2\phi)^{-1/2} \tag{75}$$

where $f = 1/298.26$ is the Earth flattening (relative difference between the equatorial and polar radius). The effect of the flattening on the timing will be at most of 1 to 2 seconds of time.

6.1 Parallax from the timings

All the numerical values in Table 5 have been computed with an assumed solar parallax $\pi_{\text{ref}} = 8.794142$, that is to say an astronomical unit of 149597870.61 km. The coefficients $A, \cdots$ are directly proportional to $\pi_\odot$.

Let two distant observers located at coordinates ϕ_1, λ_1 and ϕ_2, λ_2. Assuming that their clocks are perfectly set and that they can express their measurements in UTC, the expected contact timings of the ith contact for the first observer is,

$$t_1^{(i)} = t_{\rm geo}^{(i)} + \tilde{A}\alpha_1 + \tilde{B}\beta_1 + \tilde{C}\gamma_1 \tag{76}$$

where $t_{\rm geo}^{(i)}$ is the geocentric time of the contact. For the second observer one has a similar equation,

$$t_2^{(i)} = t_{\rm geo}^{(i)} + \tilde{A}\alpha_2 + \tilde{B}\beta_2 + \tilde{C}\gamma_2 \tag{77}$$

Therefore by subtracting these two equations one gets the expected timing difference for these two observers as,

$$\boxed{dt_{\rm C} = \tilde{A}(\alpha_2 - \alpha_1) + \tilde{B}(\beta_2 - \beta_1) + \tilde{C}(\gamma_2 - \gamma_1)} \tag{78}$$

where the coefficients are taken for the relevant contact(s) in Table 5. If they measure a difference $dt_{\rm O}$, and if the only source of discrepancy is the value of the solar parallax, then they can conclude that the true value of the solar parallax is,

$$\boxed{\pi_\odot = \pi_{\rm ref}\,\frac{dt_{\rm O}}{dt_{\rm C}}} \tag{79}$$

6.2 Parallax from the duration of the transit

6.2.1 Two observers

The comparison between observers of the duration of the transit between the second and third contact, is much easier as it requires only a good standard wristwatch at each location to assess a duration of few hours to few seconds accuracy. Therefore each observer determines a duration and can dispense with an accurate synchronisation to some form of UT. The formula for the computed value is the same as (78) with the coefficients $\tilde{A}, \ldots$ taken from the last line of Table 5. As before, if they observed a difference of duration $dt_{\rm O}$, they can conclude from this single pair of measurements a value of the parallax with Eq. 79.

It is important to stress that the reference value of the parallax is totally irrelevant and that only the precision of the observations performed at the two locations matters. Changing the reference parallax would change the coefficients $A, \cdots$ accordingly and would not affect the ratio $\pi_{\rm ref}/dt_{\rm C}$.

6.2.2 Combination of observations

Having n observers on the Earth able to observe ingress and egress provides $N = n(n-1)/2$ baselines and as many different determinations of parallaxes. Even if each observer worked

Table 9: *Topocentric coefficients for the historical Venus transits for the duration between the two inner contacts. The second column gives the computed geocentric duration between this two contacts. The reference parallax* $\pi_\odot = 8''.794142$ *has been used.*

Year	$t_3 - t_2$	$\tilde{A}$ s	$\tilde{B}$ s	$\tilde{C}$ s
1761	$05^h57^m41^s$	4.6	-313.4	-465.6
1769	$05^h42^m12^s$	476.5	376.5	516.1
1874	$03^h36^m38^s$	-293.1	-12.9	1178.6
1882	$05^h39^m03^s$	-302.1	591.9	-576.1

with the same accuracy in the timing of the contacts, the precision on the parallaxes would not be constant. A longer baseline (in fact a longer shift in the duration of the transit between the two locations) yields a more accurate parallax. Therefore one cannot combine straightforwardly the N determinations into a single one without weighing properly each determination. The weight is a function of the accuracy of the individual timings (very difficult to assess) and of the baseline. Once this is done, one must evaluate the standard error, taking into account that if there are N parallaxes, there are not N independent measurements, but only n. Thus a full $N \times N$ correlation matrix must be evaluated to assess the final standard error. The details are given in the next section with the application to the transit of 1769.

7 Application to the passage of 1769

The method developed in the previous sections can be applied to the historical transits, in particular that of 1769 when combined measurements of the inner and outer contacts have been obtained at different locations. For these transits the relevant coefficients for the topocentric durations between the 2nd and 3rd contact are given in Table 9.

Consider now the observations performed during the expeditions of 1769, restricting to the five stations where the two internal contacts of Venus with the Sun where properly seen and timed. A very simple approach consists of grouping the stations two by two so that one parallax can be estimated for each of the possible pairs. Clearly the results are not statistically independent and this would not be the method favoured today by astronomers, but basically this was the method employed in the XVIIIth century. The first two stations being very similar in position I will not combine them as a pair, although they would

Table 10: *Venus transit of June 1769 : Location and observations carried out at the five stations with timings at the inner ingress and egress. The local times are in true solar time and have not been converted into UT.*

	Place	ϕ deg	λ deg	Observer	t_2 local time	t_3 local time	$t_3 - t_2$
1	Vardö	70.367	31.02	Hell	$21^h33^m54^s$	$03^h27^m08^s$	$5^h53^m14^s$
2	Kola	68.882	33.02	Rumunsky	$21^h45^m47^s$	$03^h39^m06^s$	$5^h53^m19^s$
3	Hudson Bay	58.792	-94.26	Wales	$13^h15^m23^s$	$19^h00^m47^s$	$5^h45^m24^s$
4	St Joseph	23.060	-109.68	Chappe	$12^h17^m27^s$	$17^h54^m50^s$	$5^h37^m23^s$
5	Tahiti	-17.482	-149.48	Cook	$09^h44^m04^s$	$15^h14^m08^s$	$5^h30^m04^s$

be naturally downweighed by a proper statistical treatment. For each pair (i, j) I have computed the expected difference of duration between the inner contacts from Eq. 78 using the data of the 1769 passage in Table 9. This is then compared to the observed value and the parallax is eventually estimated from Eq. 79.

For each pair of stations I also estimated the standard error on the parallax by assuming that the timing of the inner contacts is carried out without bias (this point is obviously questionable because of the black-drop effect) and with a random error of $\tau = 10$ s in the timing. Then the error on the duration is $\sqrt{2}\tau$ and the error in the difference of duration between the station i and station j is 2τ. This leads to the standard error for the parallax based only on these two stations,

$$\sigma_\pi = \pi_{\rm ref}\ \frac{2\tau}{dt_{\rm c}} \tag{80}$$

The values are given in Table 11 and indicate a precision of the order of 0.1 to 0.4 arcsec, variable with the baseline between the two stations, as explained earlier.

To obtain a value of the parallax from these five stations one must combine the individual results. A straight mean is not appropriate, since the accuracy is not constant through the measurements. Moreover, the 9 determinations are not independent because a given station is combined with the other four. An optimal statistical combination is not trivial to find and the best compromise is to take a weighted mean between the nine determinations by assigning a weight $w(k) = 1/\sigma^2(k)$ to the kth combination. Then one computes,

Table 11: *Venus transit of June 1769 : Estimation of the solar parallax from the combination of observing stations with the timings of the 2nd and 3rd contacts. The estimate of the standard error is based on a random timing error of 10 s at each contact.*

♯	Combination	dt_{C} s	dt_{obs} s	$\pi_\odot$ ″	σ ″	AU 10^6 km
1	1 - 3	461	470	8.96	0.4	146.8
2	1 - 4	961	951	8.70	0.2	151.2
3	1 - 5	1418	1390	8.62	0.1	152.6
4	2 - 3	472	475	8.84	0.4	148.8
5	2 - 4	973	956	8.64	0.2	152.3
6	2 - 5	1430	1395	8.58	0.1	153.3
7	3 - 4	500	481	8.45	0.4	155.7
8	3 - 5	957	920	8.45	0.2	155.7
9	4 - 5	457	439	8.44	0.4	155.9

$$\pi_\odot = \sum_{k=1}^{9} w(k)\pi(k) / \sum w(k) = 8\overset{''}{.}61 \tag{81}$$

The standard error of this weighted mean can be estimated from the variance of (81), by taking into account the correlations between the different observations. It is easy to establish that the correlation coefficient $\rho(k, k')$ between the results k based on the combination from the stations (i, j) and k' for the stations (i', j') is 0, if there are no stations in common, 1/2 for the combinations (i, j) with (i, j') or (i, j) with (i', j) and -1/2 for the combinations (i, j) with (j', i) or (i, j) with (j, i'). The simple structure of the correlations matrix comes also from the uniform error in the timing ($\tau = 10$ s) assumed for each observer.

For the nine combinations of Table 11 the correlation matrix has the following form,

$$
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
\left[\begin{array}{rrrrrrrrr}
1.0 & 0.5 & 0.5 & 0.5 & 0.0 & 0.0 & -0.5 & -0.5 & 0.0 \\
0.5 & 1.0 & 0.5 & 0.0 & 0.5 & 0.0 & 0.5 & 0.0 & -0.5 \\
0.5 & 0.5 & 1.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.5 & 0.5 \\
0.5 & 0.0 & 0.0 & 1.0 & 0.5 & 0.5 & -0.5 & -0.5 & 0.0 \\
0.0 & 0.5 & 0.0 & 0.5 & 1.0 & 0.5 & 0.5 & 0.0 & -0.5 \\
0.0 & 0.0 & 0.5 & 0.5 & 0.5 & 1.0 & 0.0 & 0.5 & 0.5 \\
-0.5 & 0.5 & 0.0 & -0.5 & 0.5 & 0.0 & 1.0 & 0.5 & -0.5 \\
-0.5 & 0.0 & 0.5 & -0.5 & 0.0 & 0.5 & 0.5 & 1.0 & 0.5 \\
0.0 & -0.5 & 0.5 & 0.0 & -0.5 & 0.5 & -0.5 & 0.5 & 1.0
\end{array}\right]
$$

Then we have for the standard deviation of the weighted mean,

$$\sigma^2_{\pi_\odot} = \left(\sum_{k=1}^{9} w(k)^2 \sigma^2(k) + 2 \sum_{k} \sum_{k'>k} w(k)\, w(k')\, \rho(k,k')\, \sigma(k) \sigma(k') \right) / \left(\sum w(k) \right)^2 \quad (82)$$

yielding the final result,

$$\boxed{\pi_\odot = 8.61 \pm 0.10} \quad (83)$$

(If the correlations are omitted the standard error is only $0''.06$). This result is in good agreement with the extensive and numerous discussions that followed the transit of 1761, whose final conclusion was a solar parallax between $8''.45$ and $8''.85$ and the re-evaluations on the XIXth century by Encke in 1824 leading to $\pi_\odot = 8''.6030$.

8 Worked example for the passage of 2004

This section summarises the results developed in detail in the course of this paper, applying the main formulas to the passage of 2004. Its intent is very practical and directed to users who wish to skip the rather intricate mathematical derivation. It can be read almost independently of the rest of the text.

The difference δt of time between the contacts observed at the centre of the Earth and at a point of longitude λ and latitude ϕ at its surface is computed by the following formula,

$$\delta t = A\, \cos\phi\, \cos\lambda + B\, \cos\phi \sin\lambda + C\, \sin\phi \quad (84)$$

where the coefficients A, B, C are numerical constant independent of the observer and computed once for all for the transits, for each contact. Their values in seconds of time are given in Table 12. The accuracy of this approximation to obtain the topocentric contacts is estimated to about 5 seconds, and is considered sufficient to analyse most of the observations of contact timings.

Table 12: *Venus transit of June 2004 : Geocentric contacts and Coefficients A, B, C giving the time shift of the topocentric contacts relative to the corresponding geocentric ones.*

Contact	Time UT	A s	B s	C s
1st	$05^h13^m34^s$	388.6	4.9	174.4
2nd	$05^h32^m50^s$	396.5	-38.6	202.9
3rd	$11^h06^m41^s$	195.5	-206.0	-345.3
4th	$11^h25^m57^s$	166.9	-230.7	-316.8

For the duration of the transits between the outer or inner contacts, the difference between the two contacts determined with (84) yields an expression of the same form, with different numerical factors. They are given in Table 13 together with the geocentric duration. These coefficients have been computed with a reference value for the parallax $\pi_{\rm ref} = 8''.794142$ and are directly proportional to the actual parallax. Any other value could have been used and this has absolutely no impact at all on the determination of the Astronomical Unit that a network of observers would obtain from their own timings of the transit.

Table 13: *Venus transit of June 2004 : Duration of the transits for the outer and inner contacts and coefficients A, B, C needed to compute the corresponding topocentric durations.*

Contact	$\delta t_{\rm geo}$ mn	A s	B s	C s
1st to 4th	372.40	-221.6	-235.7	-491.2
2nd to 3rd	333.85	-200.9	-167.4	-548.2

Two observers measuring each at its location the duration of the transit between the two inner contacts can determine the solar parallax from their measurements with (78)and (79), by computing the expected difference of duration of the transits between the two places from the reference distance, and using the observed value in (79). A similar formula can be written giving directly the astronomical unit in kilometres as,

$$\boxed{1\mathrm{AU} = 149.60 \times 10^6 \, \frac{\delta t_{\mathrm{ref}}}{\delta t_{\mathrm{obs}}}} \tag{85}$$

As a numerical application consider an observer located at Nice ($O_1 : \lambda = 7.^\circ 30, \phi = 43.^\circ 72$) and the second at Saint-Denis in the Island of Reunion ($O_2 : \lambda = 55.^\circ 47, \phi = -20.^\circ 87$). One determines first the computed duration (between second and third contact) for each place with (84), taking the coefficients A, B, C and the geocentric duration from the second line of Table 13. One gets,

$$\delta t_1 = \delta t_{\mathrm{geo}} - 538.3\,\mathrm{s} = 324.87\,\mathrm{mn} \tag{86}$$

$$\delta t_2 = \delta t_{\mathrm{geo}} - 40.0\,\mathrm{s} = 333.18\,\mathrm{mn} \tag{87}$$

for the duration of the transit at Nice and at Saint-Denis. The differences in the duration between the geocentric event and the topocentric event expressed in seconds follow just from a straight application of (84) to the two locations. Therefore the theoretical difference between the two measurements computed with a reference astronomical unit of 149.60×10^6 km should be,

$$\delta t_{\mathrm{ref}} = 498.3\,\mathrm{s} \tag{88}$$

Assuming that the observed values for the duration are (with a reasonable random error in the timing of about 20 s),

$$\delta t_1 = 324.60\,\mathrm{mn} \tag{89}$$

$$\delta t_2 = 333.30\,\mathrm{mn} \tag{90}$$

or

$$\delta t_{\mathrm{obs}} = 522.0\,\mathrm{s} \tag{91}$$

Therefore, they would find with (85) an astronomical unit of 142.9×10^6 km, or an error of $\sim 5\%$.

It happens that the linear formula performs poorly in the middle of the Indian Ocean (see Fig. 7-bottom) and that in this particular case, a processing with the quadratic expression would be more appropriate. Essentially, one has to correct the observed duration for the magnitude of the second order terms before applying the result to the determination of the parallax with the linear terms.

ANNEX

Numerical data for the transit of June 2012

The following tables are identical to Table 4-5-6, but for the transit of 2012 with an extrapolated value of $TT - TU = 65.8$s.

Table 14: *Venus transit of 5-6 June 2012 – Geocentric contacts*

Contact	Time	X	Y	$\dot{X}$	$\dot{Y}$	θ	$H_\odot$
	UT	$''$	$''$	$''$/h	$''$/h	$''$	deg
1st	$22^h09^m44^s$	633.84	738.38	-232.58	-60.92	945.72	152.78
2nd	$22^h27^m33^s$	564.83	720.31	-232.58	-60.92	945.72	157.23
3rd	$04^h31^m45^s$	-847.59	353.06	-232.78	-60.07	945.69	248.27
4th	$04^h49^m34^s$	-916.66	335.23	-232.78	-60.07	945.69	252.72

Table 15: *Venus transit of 5-6 June 2012. Coefficients for the topocentric contacts : linear formula.*

Contact	$A + \delta A$ s	$B + \delta B$ s	$C + \delta C$ s	Γ s	b deg	l deg
1st	-222.7	176.1	-277.3	396.9	-44.3	141.7
2nd	-213.9	184.4	-296.9	409.8	-46.4	139.2
3rd	373.9	82.0	144.5	409.1	20.7	12.3
4th	371.4	59.0	125.0	396.3	18.4	9.0
2nd to 3rd	587.8	-102.5	441.4	742.2	36.5	350.1
1st to 4th	594.1	-117.1	402.2	727.0	33.6	348.9

Table 16: *Venus transit of 5-6 June 2012. Coefficients for the topocentric contacts : second order terms.*

Contact	C_{00} s	C_{22} s	S_{22} s	C_{21} s	S_{21} s	C_{20} s
1st	2.2	-1.1	-0.0	-0.4	0.1	3.0
2nd	2.6	-1.1	0.1	-0.2	0.2	3.6
3rd	-2.6	0.7	-1.4	-1.4	-0.2	-3.6
4th	-2.2	0.5	-1.4	-1.2	-0.0	-3.0
2nd to 3rd	-5.1	1.7	-1.5	-1.2	-0.4	-7.1
1st to 4th	-4.4	1.6	-1.4	-0.8	-0.1	-6.1

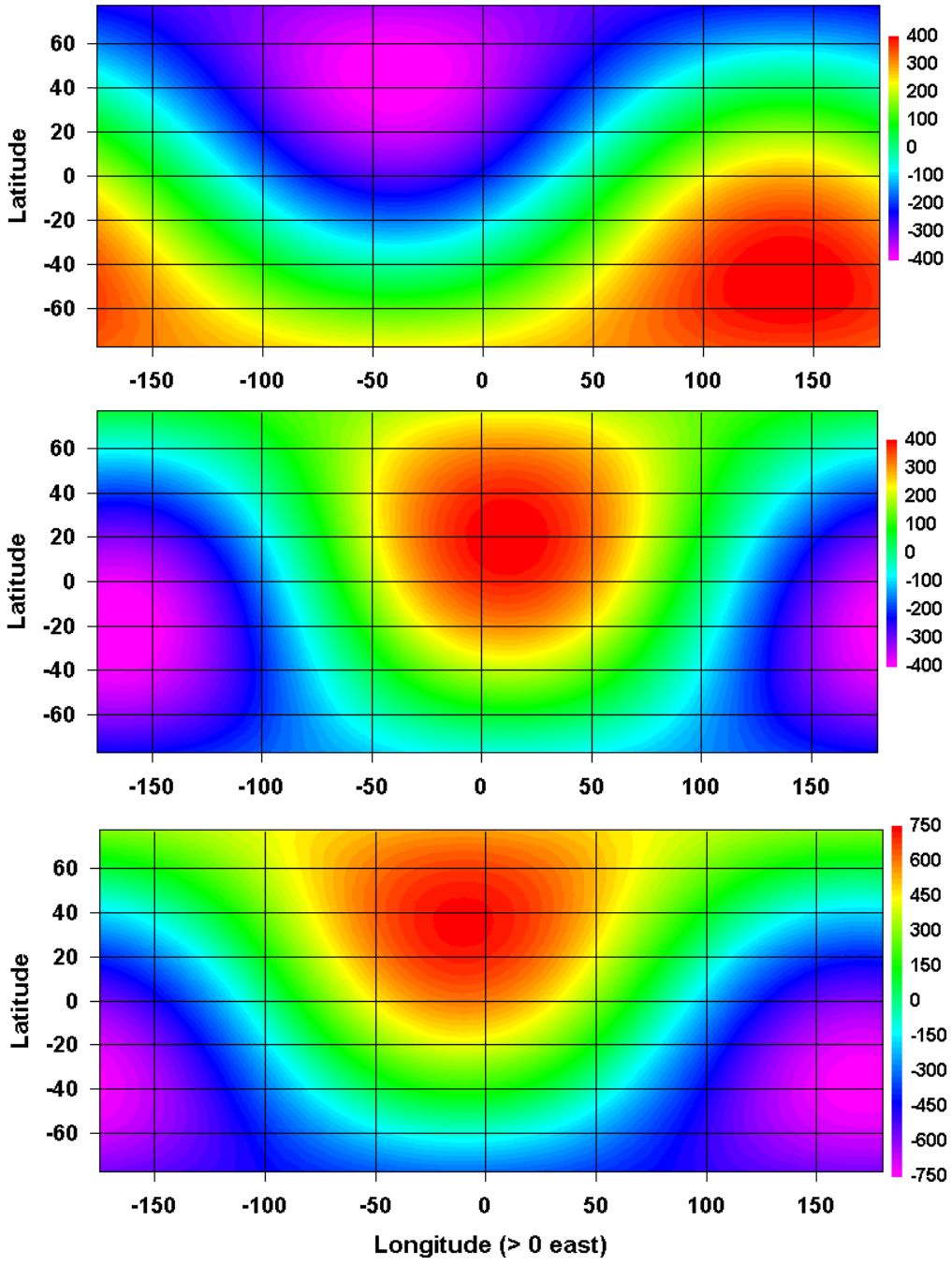


Figure 9: *Difference in seconds between the geocentric and topocentric time of the second (top) and third contact (middle), and difference of duration of the transit (bottom) for any point on the Earth (irrespective of the visibility)*